Gravitational collapse to extremal black holes and the third law of black hole thermodynamics

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joint work with Ryan Unger (Princeton)

BLACK HOLE THERMODYNAMICS

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics is a proposed close mathematical **analogy** between black hole dynamics and classical thermodynamics.

Black hole thermodynamics

Law	Classical thermodynamics	Black holes

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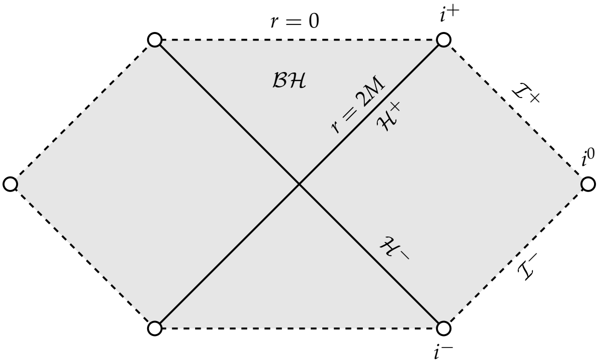
- Proposed by Bardeen–Carter–Hawking '73 as statements within **classical GR**.
- Hawking radiation—interpret $\kappa \propto T$
- Bekenstein entropy—interpret $A \propto S$
- $\kappa \propto T$, $A \propto S$ further justified once quantum effects are considered, but they are **formal analogies**.
- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Refresher on Schwarzschild



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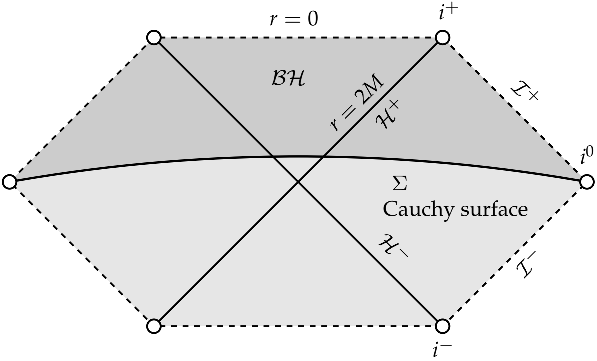
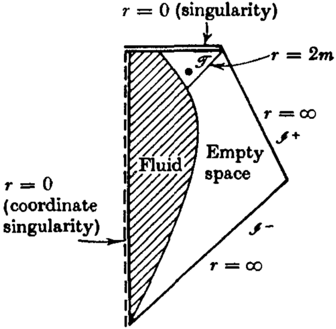
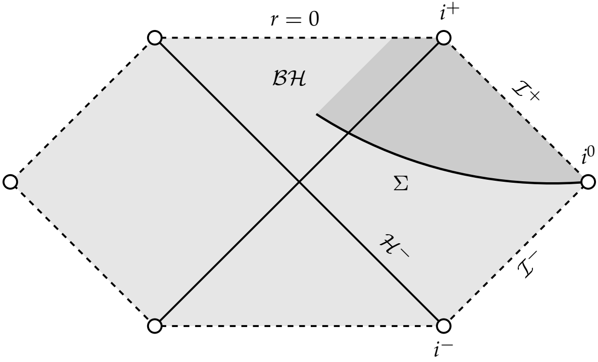


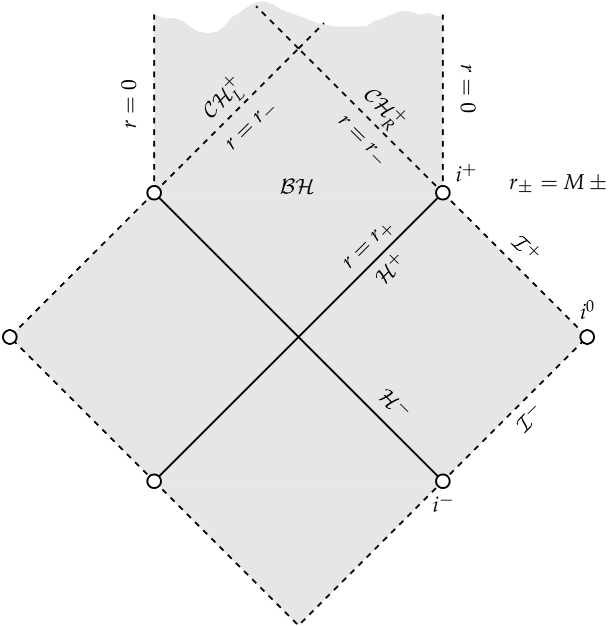
Maximally extended Schwarzschild is the unique maximal Cauchy development of the data induced on a spacelike hypersurface $\Sigma \cong \mathbb{R} \times S^2$.



Penrose diagram of gravitational collapse. One-ended Cauchy data!


$r = 0$
i^+
$\mathcal{BH}$
$\mathcal{I}^+$
i^0
Σ
$\mathcal{H}^-$
$\mathcal{I}^-$
i^-

Refresher on subextremal Reissner–Nordström: $0 < |e| < M$



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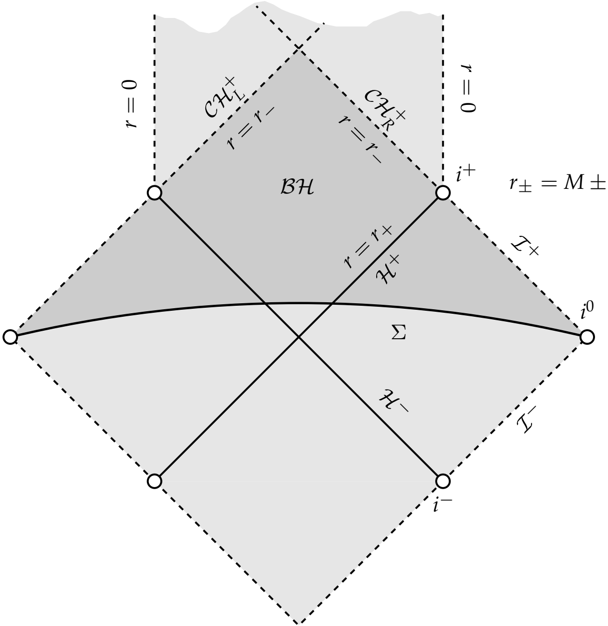

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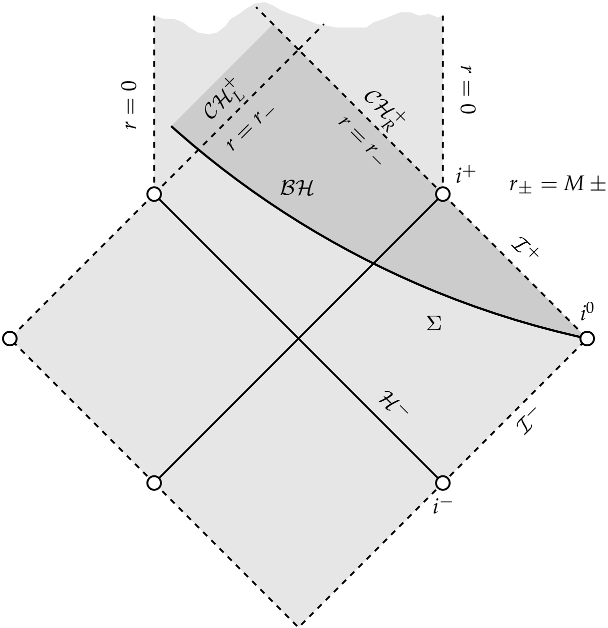

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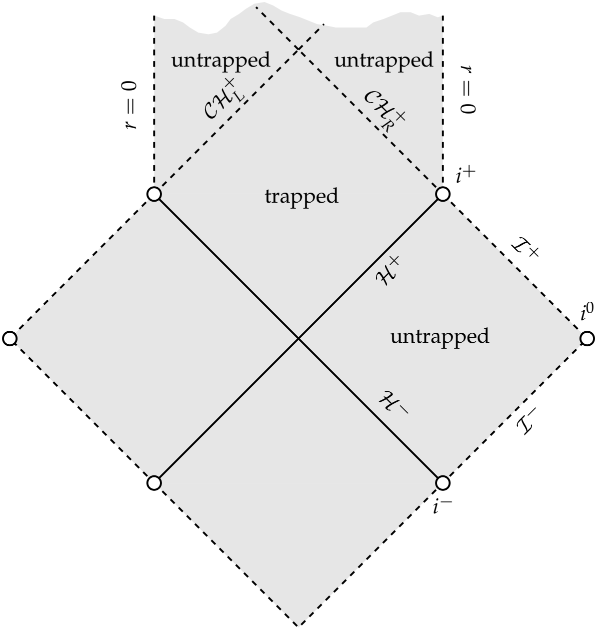

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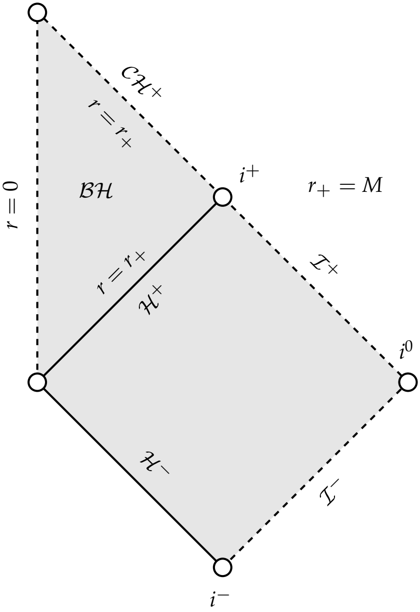

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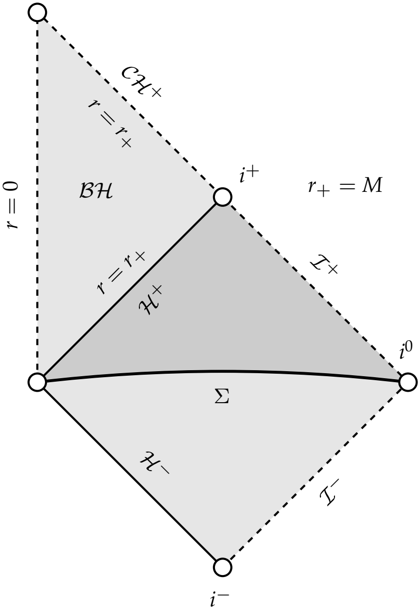

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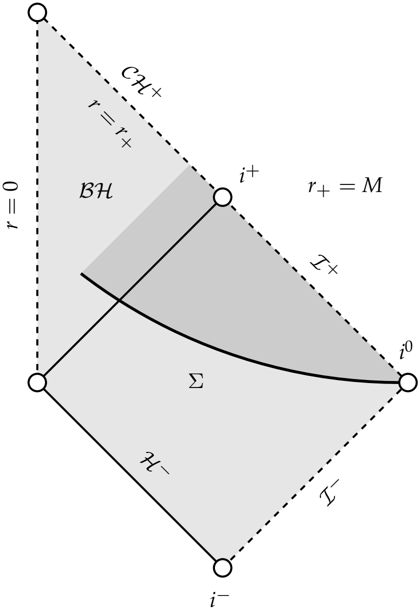

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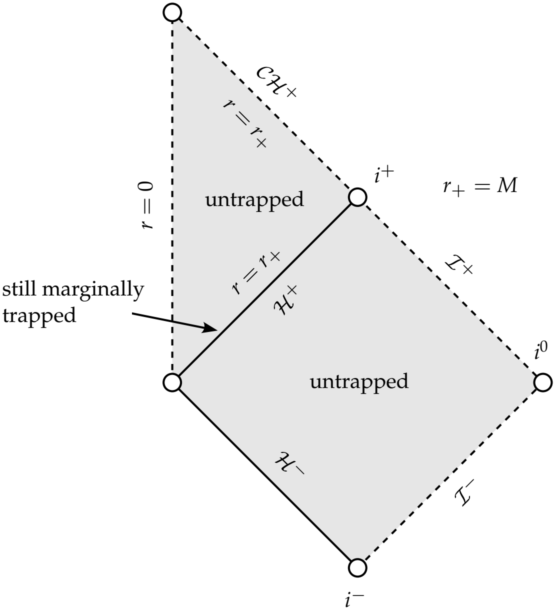
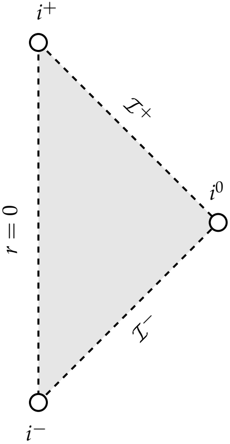


Refresher on superextremal Reissner–Nordström: $0 < M < |e|$



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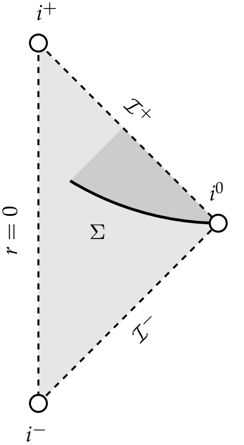

Surface gravity κ of Reissner–Nordström

- RN with mass M and charge e, $|e| \leq M$, has

$$\kappa = 2\pi T = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- **Subextremal:** $\kappa > 0$
- **Extremal:** $\kappa = 0$

Original formulation of Bardeen–Carter–Hawking:

The Third Law

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W. Israel[a]
Research Institute for Fundamental Physics, Yukawa Hall, Kyoto University, Kyoto 606, Japan
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4. **Weak energy condition** must be enforced.
   - Otherwise: counterexample using charged null dust. (Sullivan–Israel '80)

Conjecture (The third law, BCH '73, Israel '86).

A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

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Theorem (K.–Unger '22).

Subextremal black holes can become extremal in finite time, evolving from regular Cauchy data for the Einstein–Maxwell-charged scalar field system.

In particular, the "third law of black hole thermodynamics" is false.

Israel's argument I

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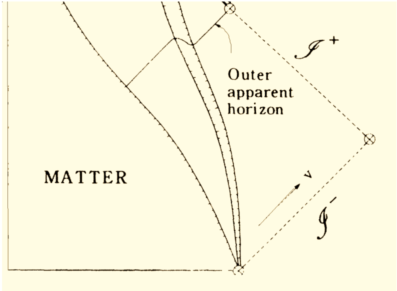
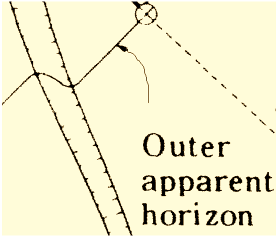


- First incoming matter flux creates (dynamical) subextremal apparent horizon.
- Second matter flux pushes to extremal horizon.

ISRAEL'S ARGUMENT II



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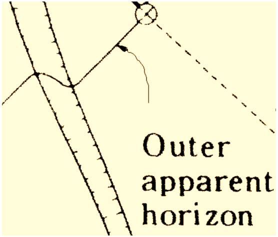

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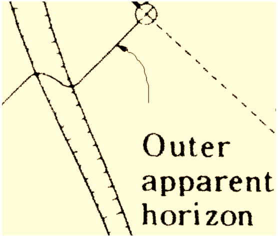

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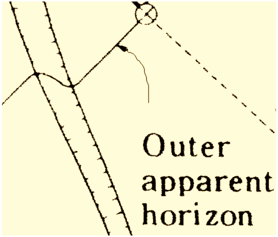

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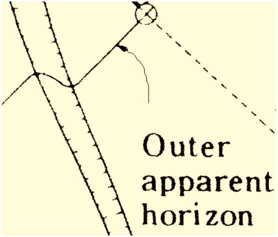

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However, outer apparent horizon can jump in smooth spacetimes.

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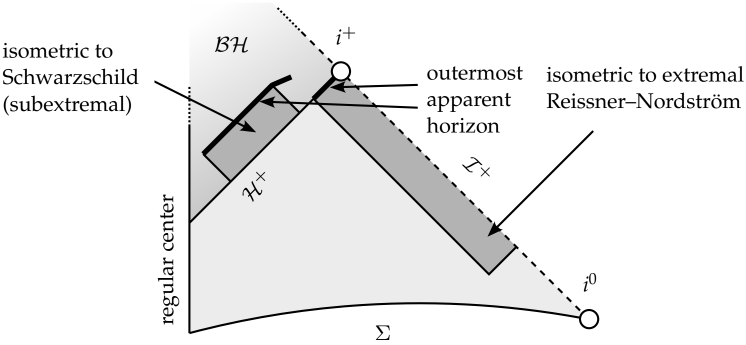
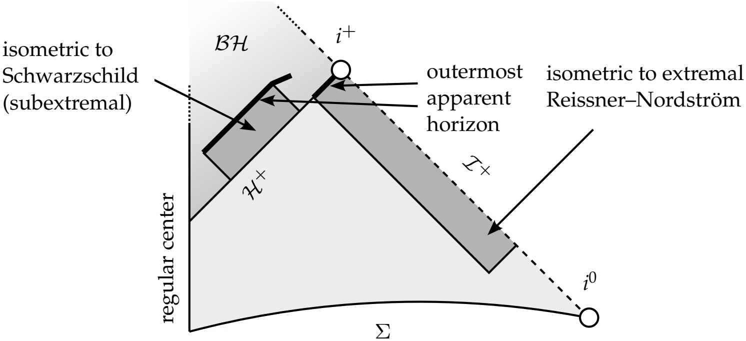


- Fine-tuned Cauchy data on Σ for Einstein–Maxwell-charged scalar field which undergo gravitational collapse.



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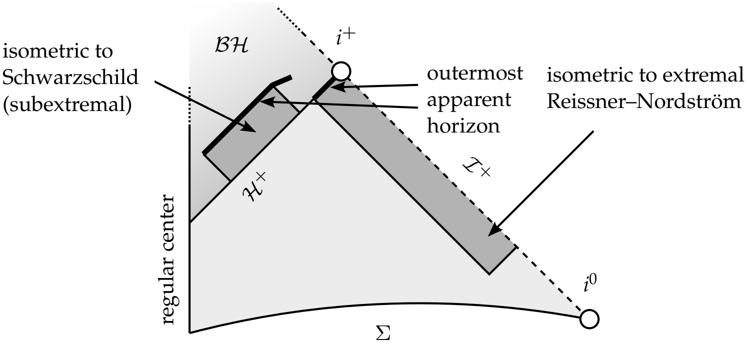

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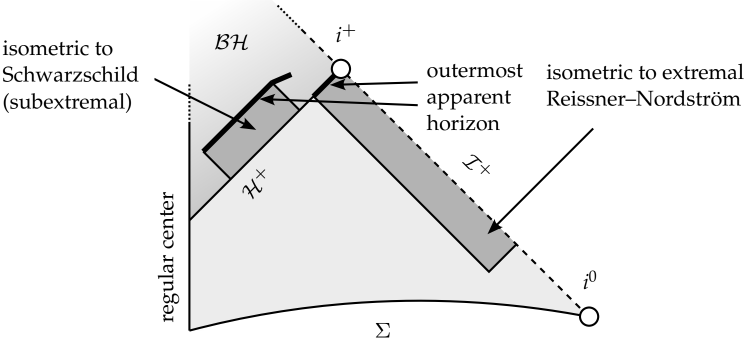

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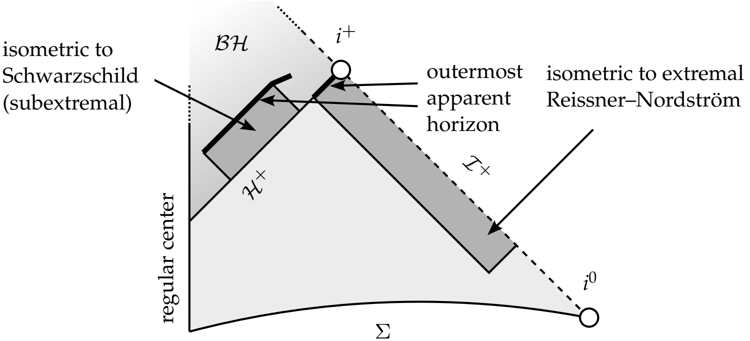

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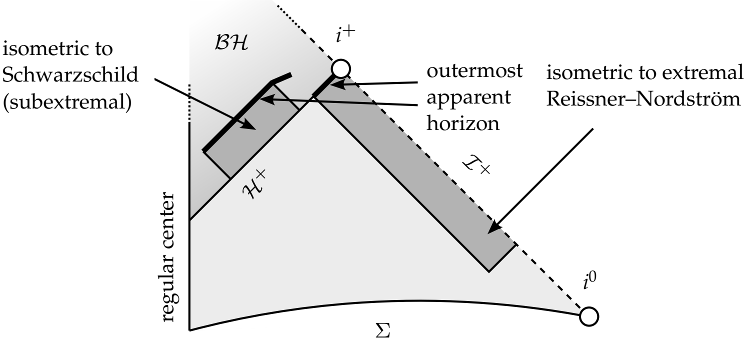

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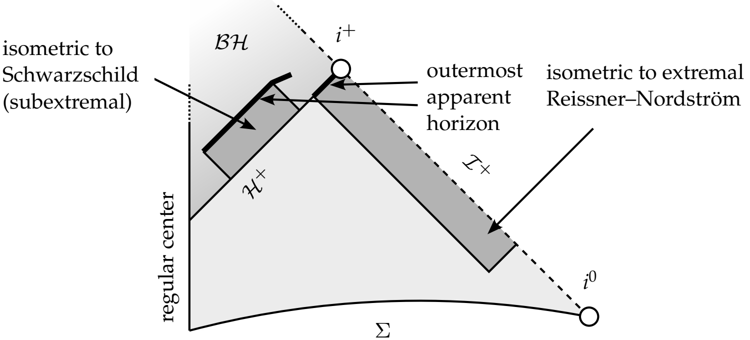

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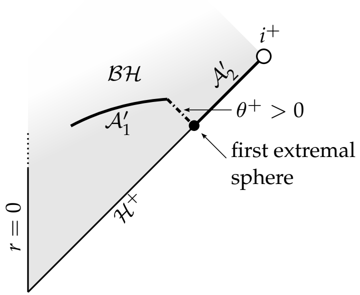
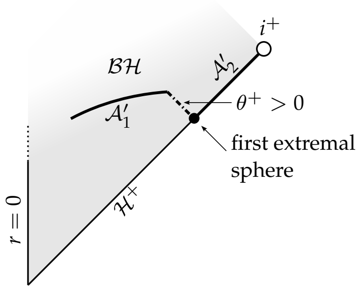
- Cauchy data are fined-tuned, yet "∞-dimensional".
- Proof directly extends to massive fields for $\mathfrak{m} \ll |\mathfrak{e}|$.



Outermost apparent horizon becomes disconnected the instant the black hole becomes extremal!



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This is a feature, not a bug!

Einstein–Maxwell-charged scalar field system

- Lorentzian manifold $(\mathcal{M}^{3+1}, g)$
- 2-form $F = dA$ (electromagnetism)
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g^{\mu\nu}D_\mu D_\nu\phi &= 0 \\
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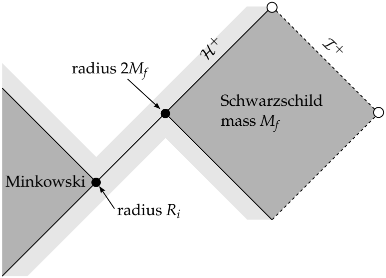
- Spherical symmetry!

Prototype: Minkowski to Schwarzschild gluing



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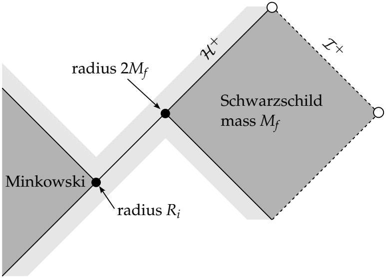

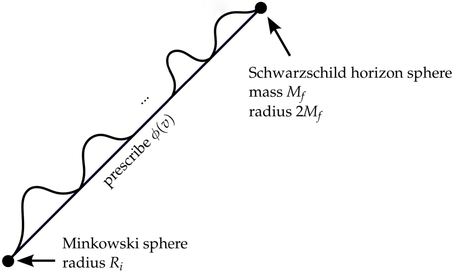

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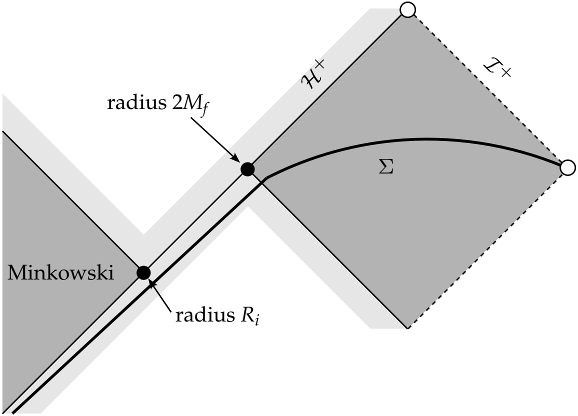
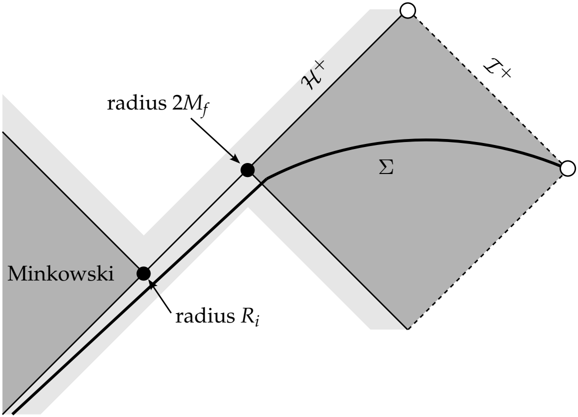
- **Enemy:** Decoherence of waves.
- **Solution:** Characteristic gluing makes superposition of waves **purely ingoing** along a single outgoing null hypersurface, e.g. $\mathcal{H}^+$

Characteristic data to Cauchy data



Obtain **Cauchy data** by solving backwards.

PROTOTYPE: MINKOWSKI TO SCHWARZSCHILD GLUING



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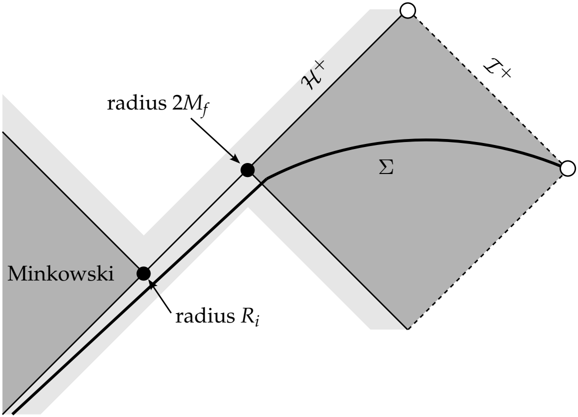
Prototype: Minkowski to Schwarzschild gluing



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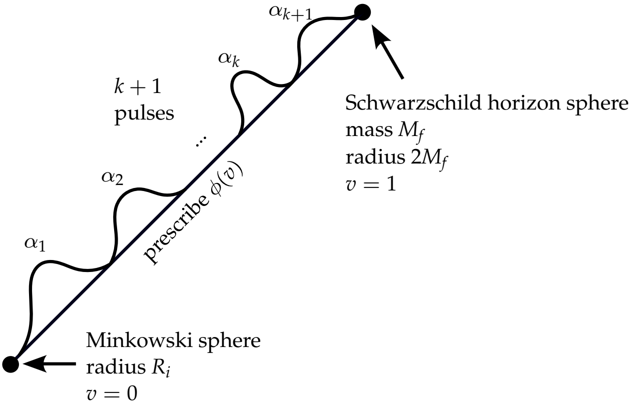
The present problem is fundamentally **nonlinear** and **nonperturbative**, but is tractable because of the symmetry assumption.

Theorem (K–Unger '22).
For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.



Aspects of the proof: Schwarzschild

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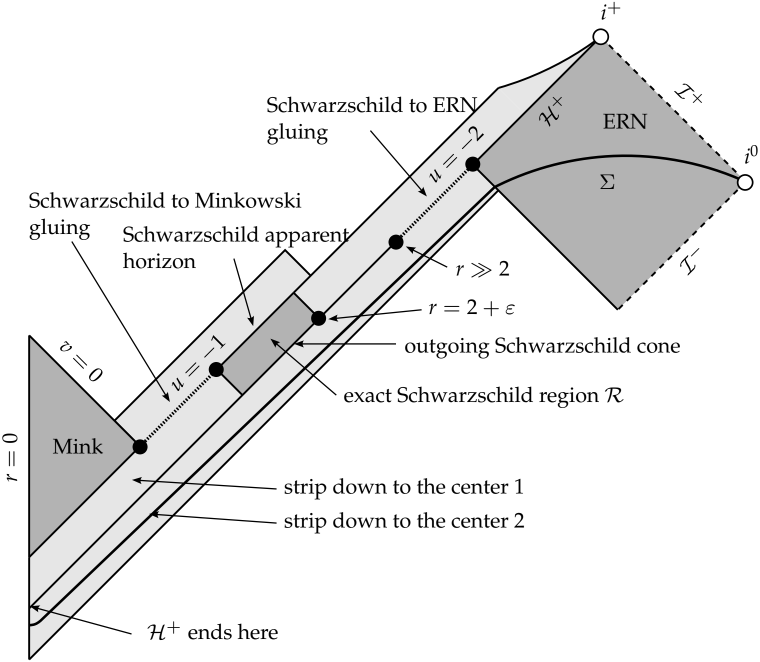
- Charged scalar field: sphere of α's is deformed topological sphere ensuring the "charge" condition.


i^+
Schwarzschild to ERN
gluing
$\mathcal{H}^+$
ERN
$\mathcal{I}^+$
$u = -2$
i^0
Σ
Schwarzschild to Minkowski
gluing
Schwarzschild apparent
horizon
$r \gg 2$
$\mathcal{I}^-$
$r = 2 + \varepsilon$
$v = 0$
$u = -1$
outgoing Schwarzschild cone
exact Schwarzschild region $\mathcal{R}$
$r = 0$
Mink
strip down to the center 1
strip down to the center 2
$\mathcal{H}^+$ ends here

Conjecture.
There exist Cauchy data for the Einstein vacuum equations

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Theorem (K.–Unger, '23).
For any $0 \leq |a| \ll M$, there exist Cauchy data for the Einstein vacuum equations

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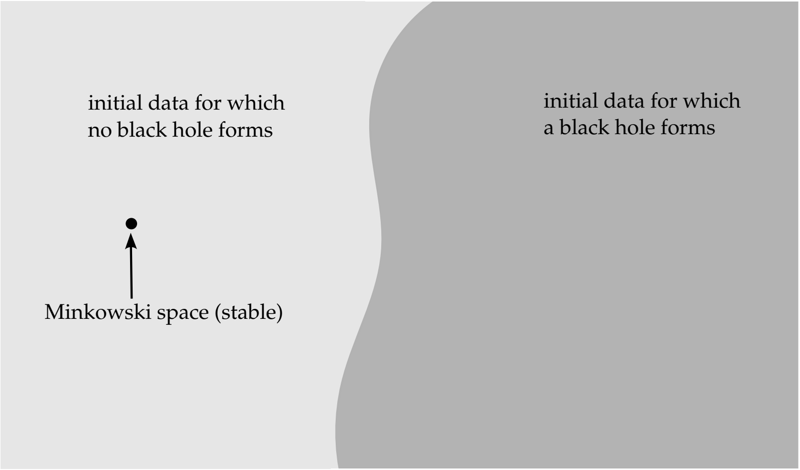
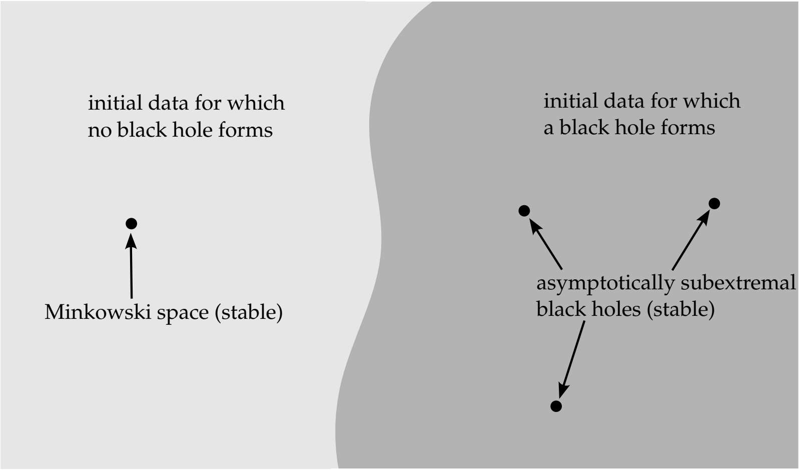


Outlook: "moduli space" of gravitational (non-)collapse



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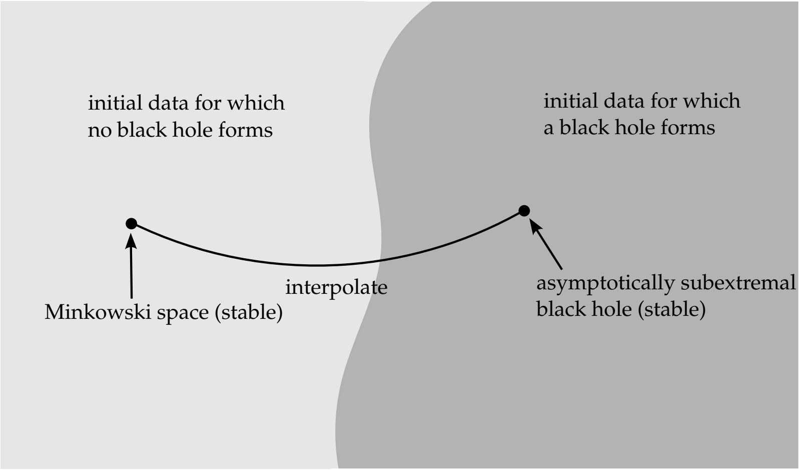

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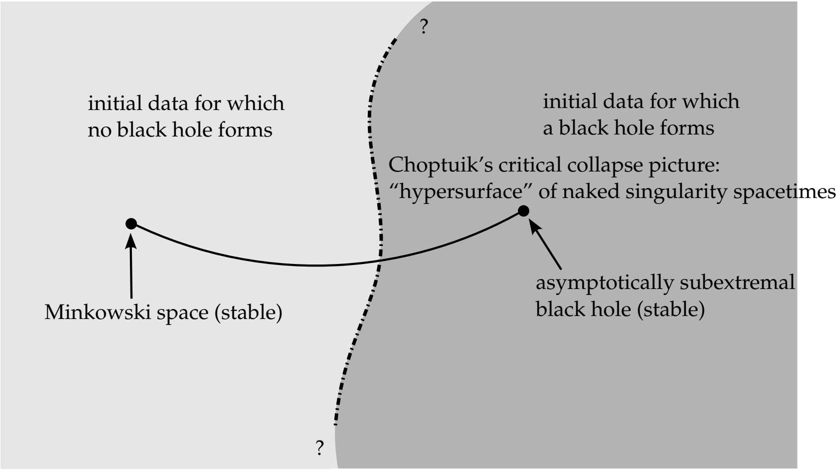

<u>Additional critical behavior:</u> Some extremal black holes lie at the threshold!

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We propose to first study this phenomenon in the Einstein–Maxwell–massless Vlasov model—inspired by classical work of A. Ori.

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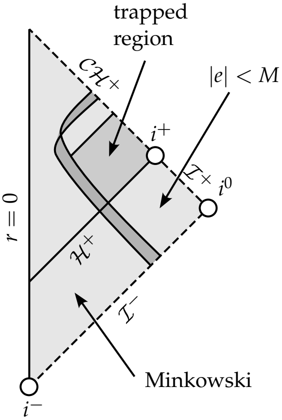
$\lambda > 0$: **subextremal** BH forms



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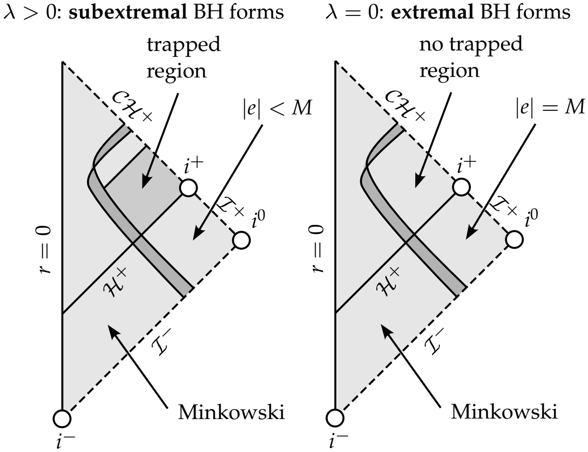

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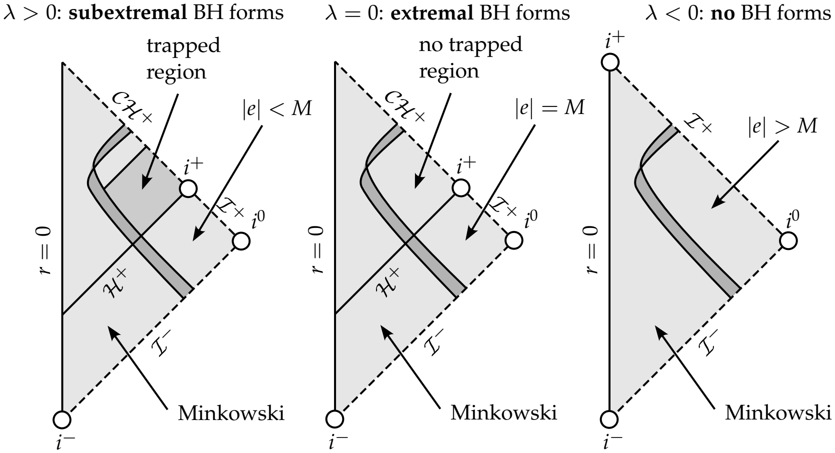

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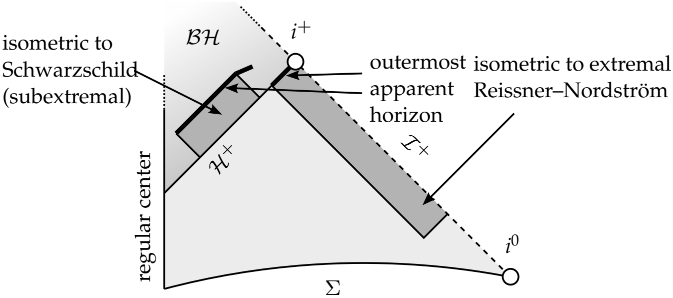

arXiv:2211.15742