Steady state holographic turbulence

A. Yarom together with Y. Oz and S. Waeber

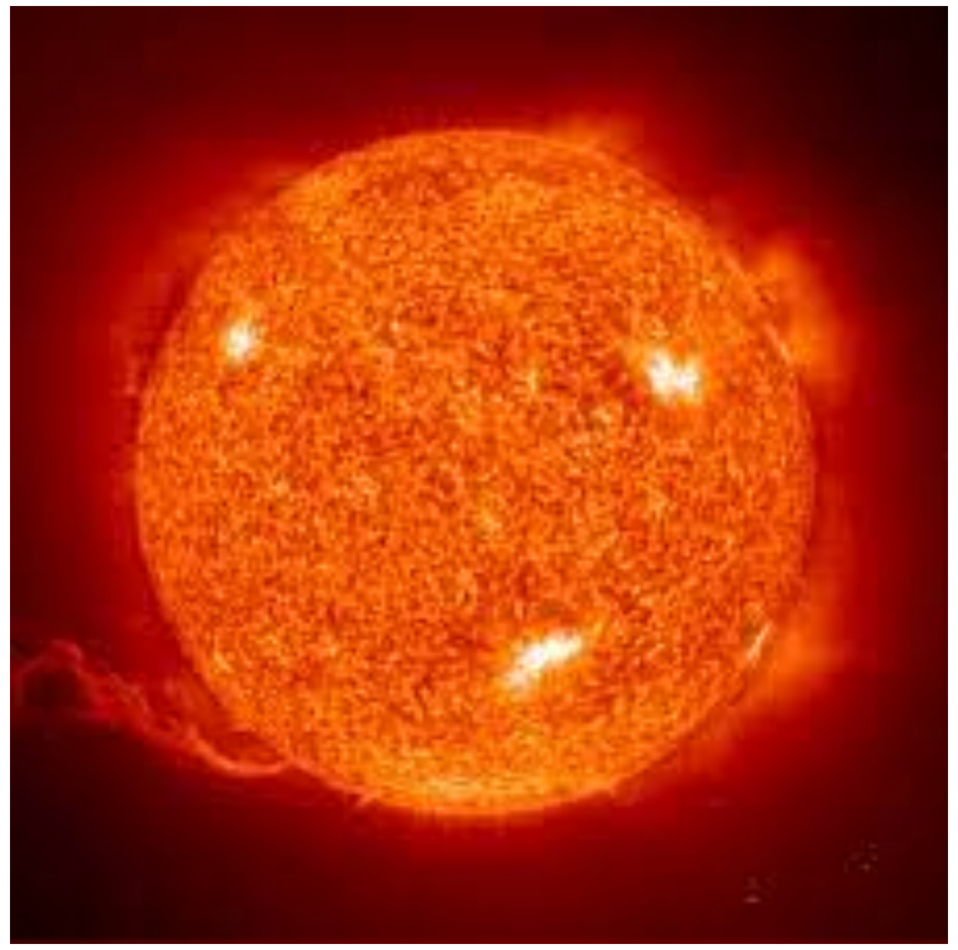
Turbulence

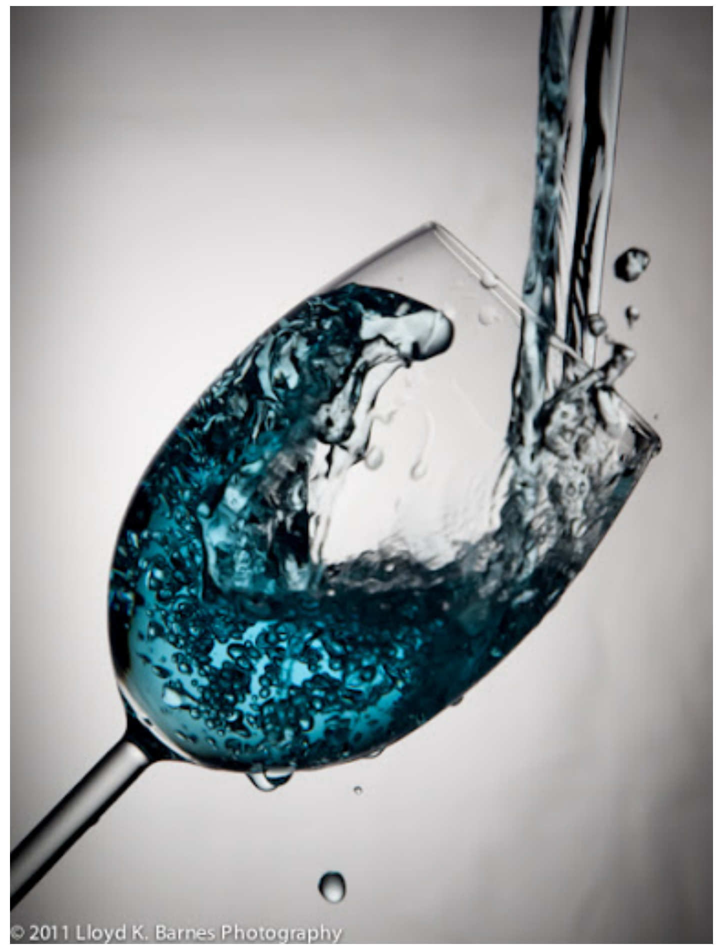
Recall:

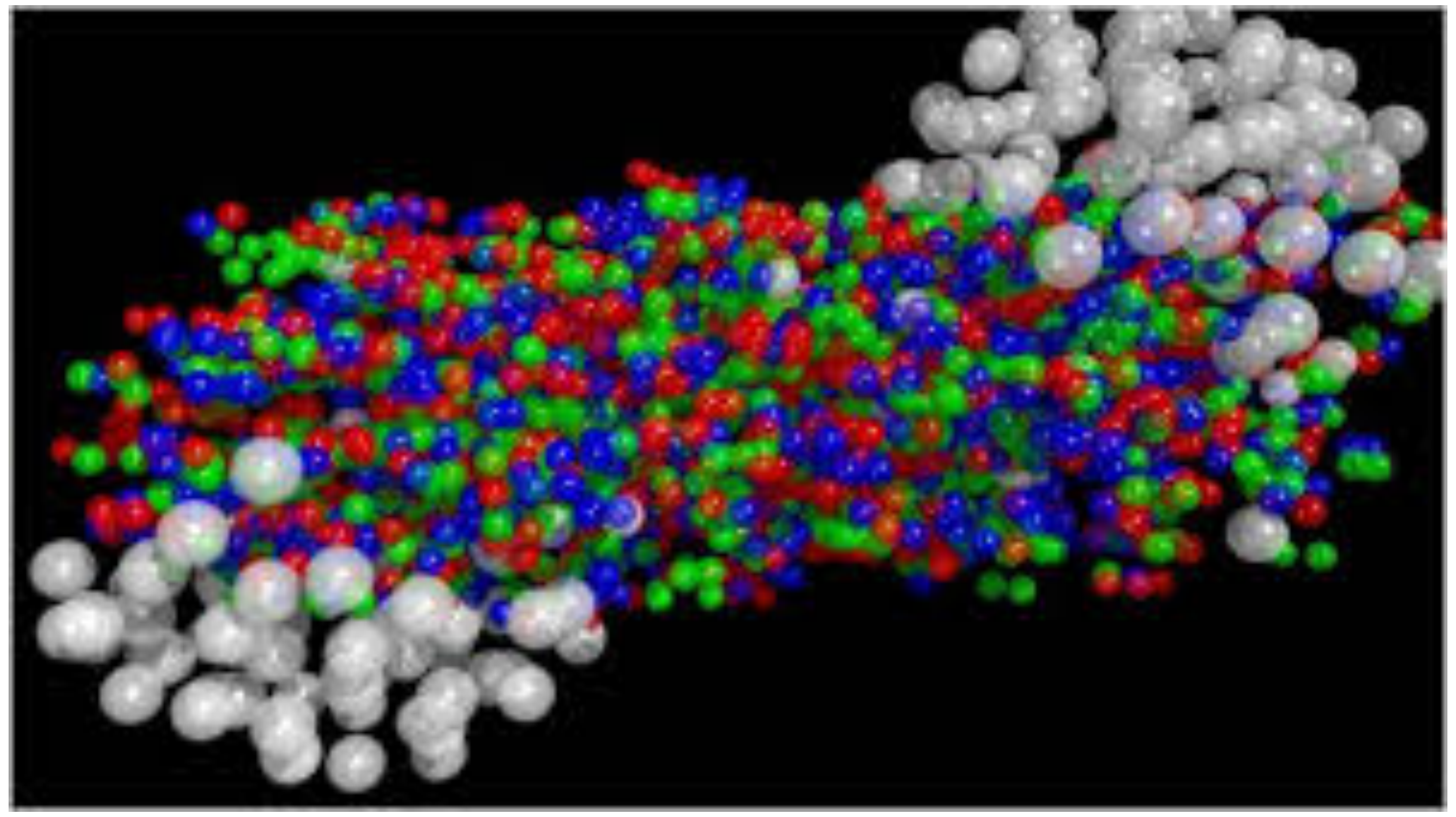
$$\dot{\vec{v}} + \vec{v} \cdot \vec{\nabla} \vec{v} = - \vec{\nabla} p + \nu \nabla^2 \vec{v} + \vec{f}$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

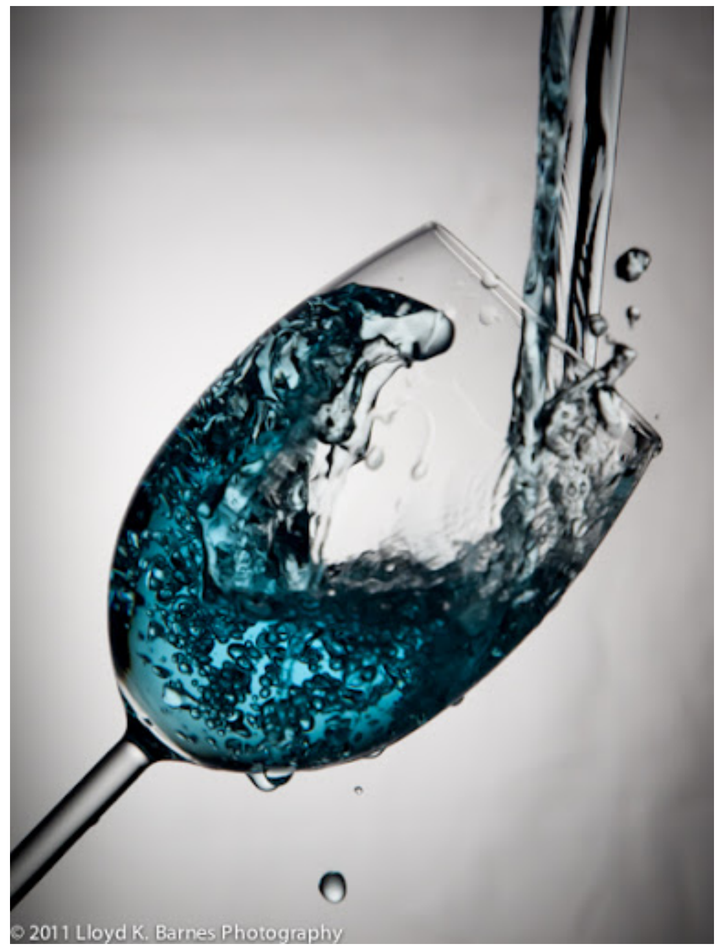
The Navier Stokes equations describe a multitude of phenomenon:







Turbulence



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$$\vec{\nabla} \cdot \vec{v} = 0$$

One characteristic of turbulence is the scaling behaviour of the kinetic energy (per unit mass).

Define

$\hat{\epsilon}(k)dk$ -Amount of kinetic energy between k and $k + dk$

Then

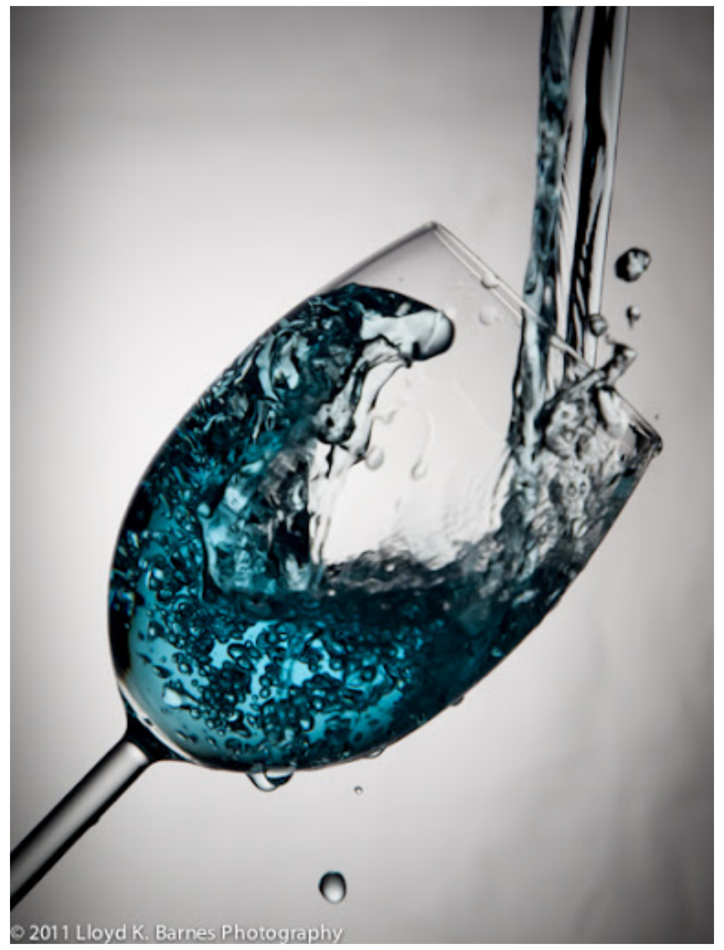
$$\bar{\hat{\epsilon}} \propto k^{-5/3}$$

Turbulence

Recall:

$$\dot{\vec{v}} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p + \nu \nabla^2 \vec{v} + \vec{f}(k_f) \qquad \overline{\vec{f}(k_f)} = 0$$

$$\vec{\nabla} \cdot \vec{v} = 0$$



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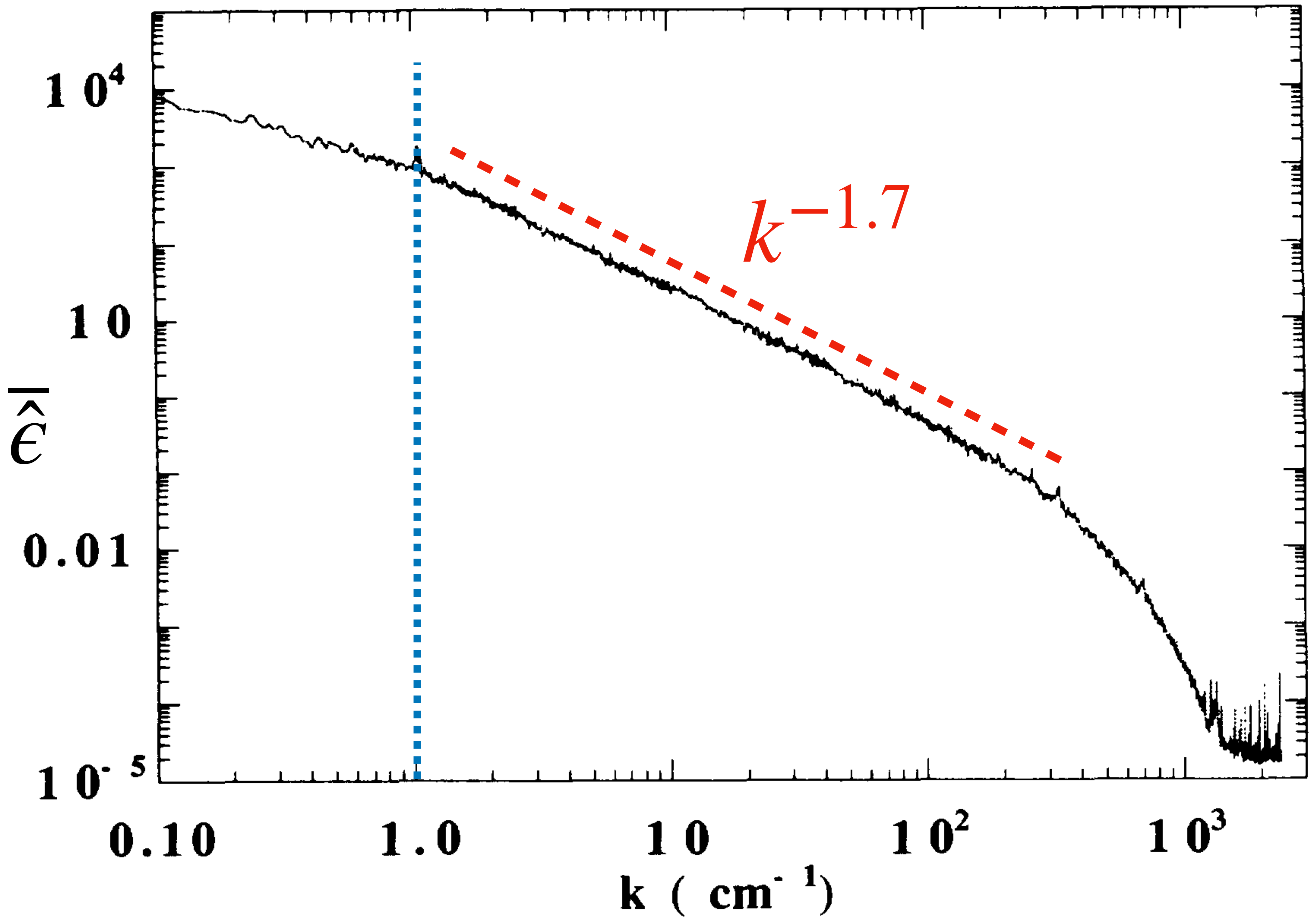

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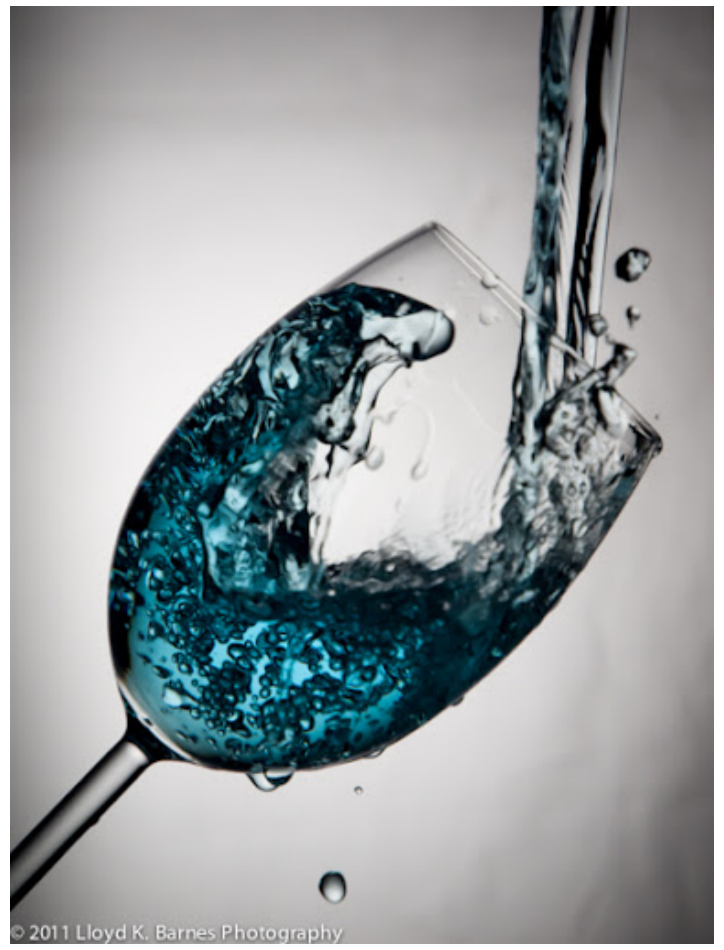


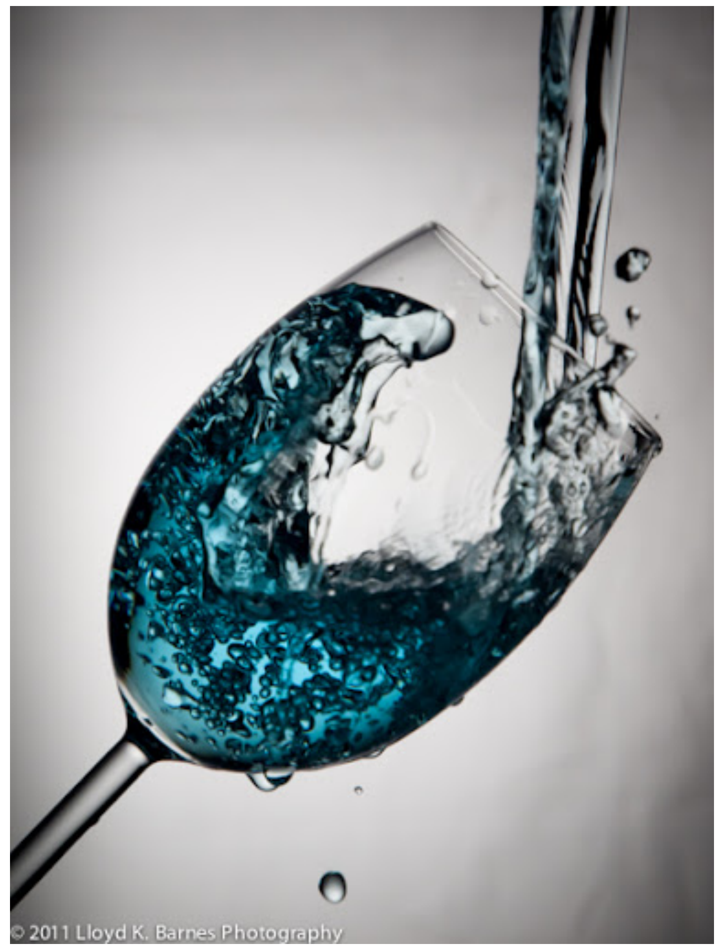
One characteristic of turbulence is the scaling behaviour of the kinetic energy (per unit mass).

$$\overline{\hat{\epsilon}} \propto k^{-5/3} \qquad\qquad (n = 2)$$

This is part of a broader set of predictions:

$$\overline{\left((\vec{v}(\vec{r}) - \vec{v}(0)) \cdot \hat{r}\right)^n} \propto |r|^{\frac{n}{3}}$$

Turbulence



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$$\vec{\nabla} \cdot \vec{v} = 0$$

A broad set of predictions

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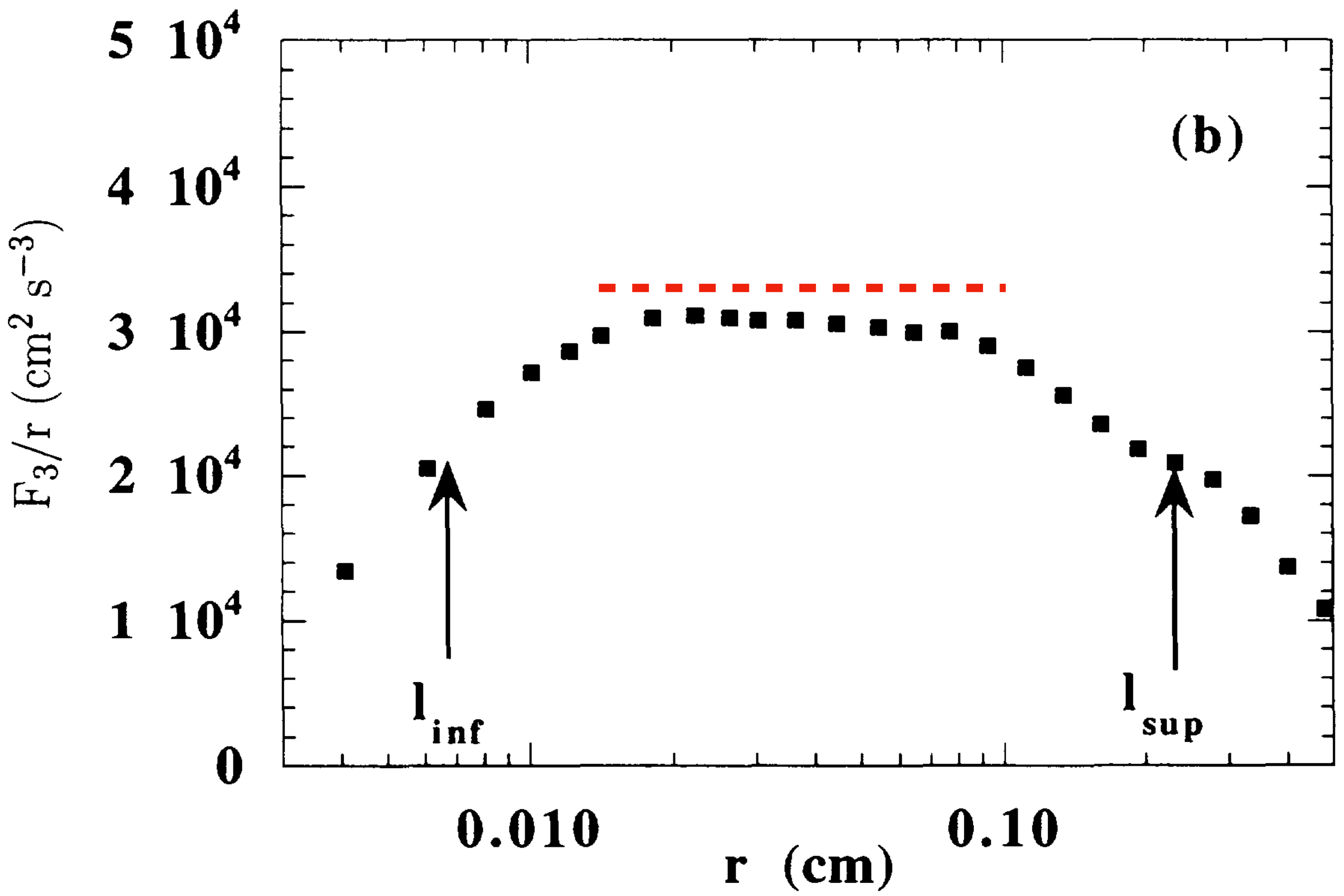
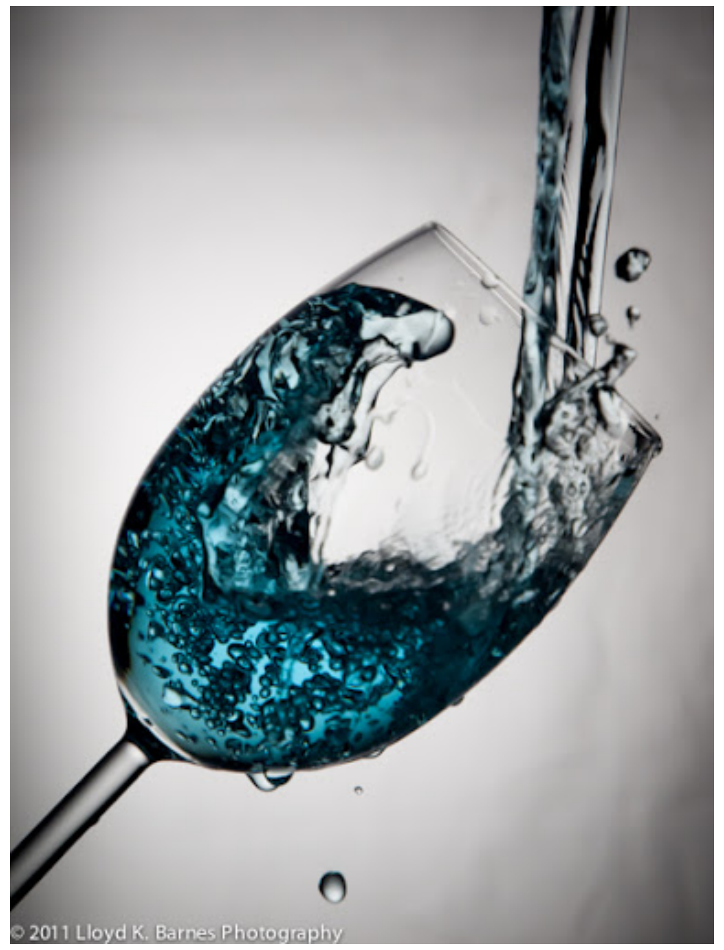
$$\overline{\left((\vec{v}(\vec{r}) - \vec{v}(0)) \cdot \hat{r} \right)^n} \propto |r|^{\frac{n}{3}}$$

For n=3,

$$F_3 = \overline{\left((\vec{v}(\vec{r}) - \vec{v}(0)) \cdot \hat{r} \right)^3} \propto |r|$$



Turbulence



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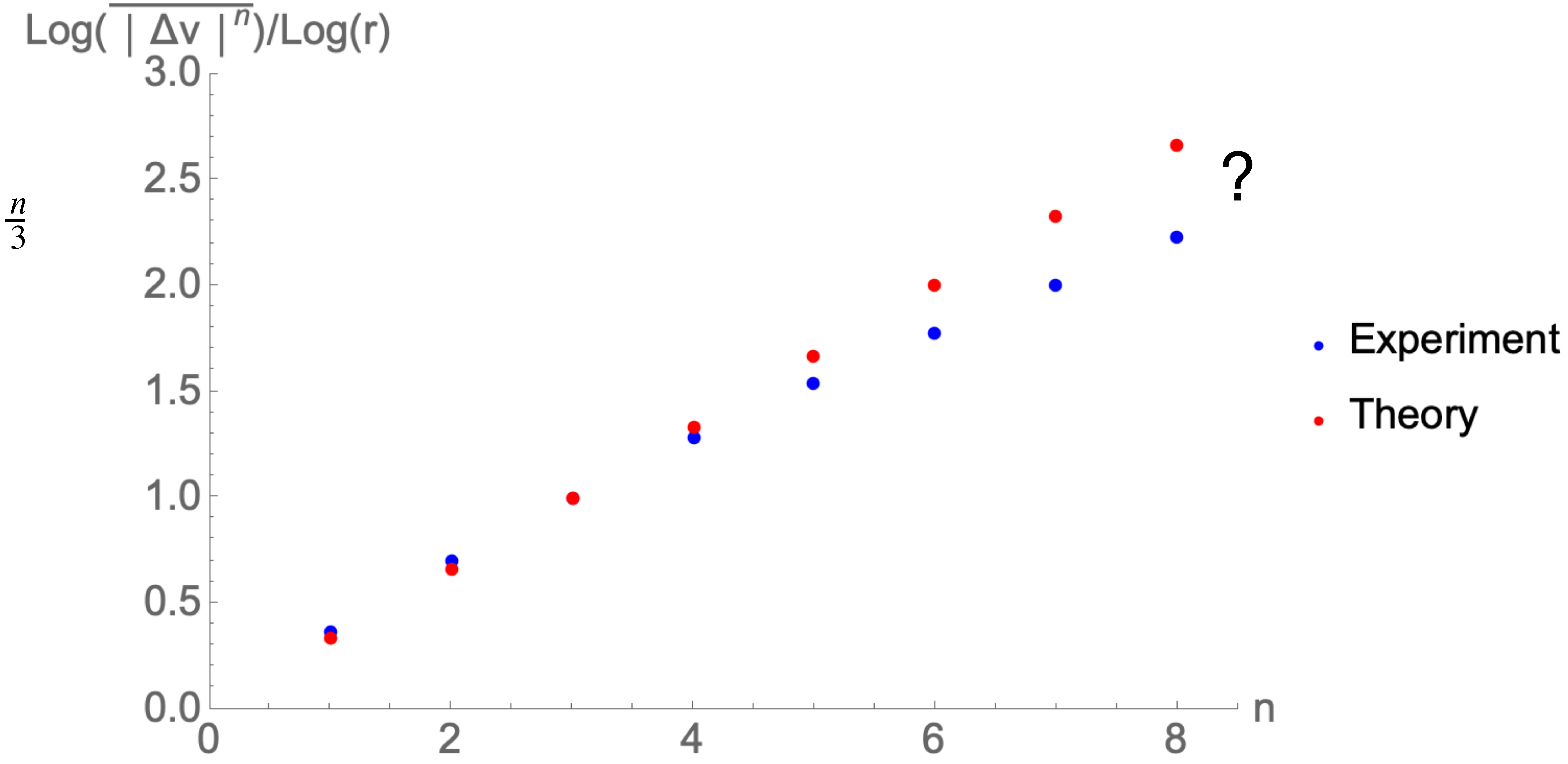
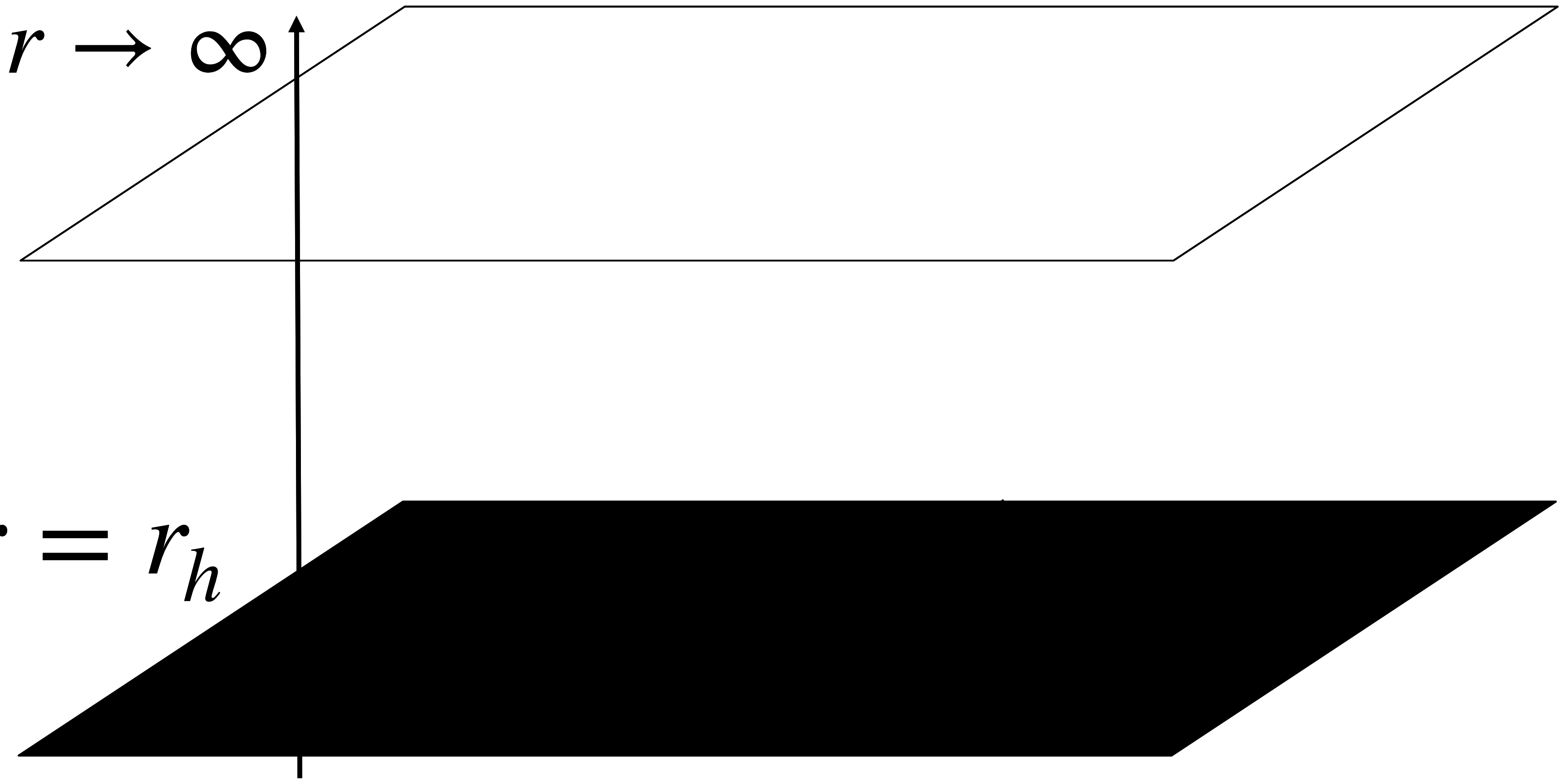


Holographic turbulence

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$



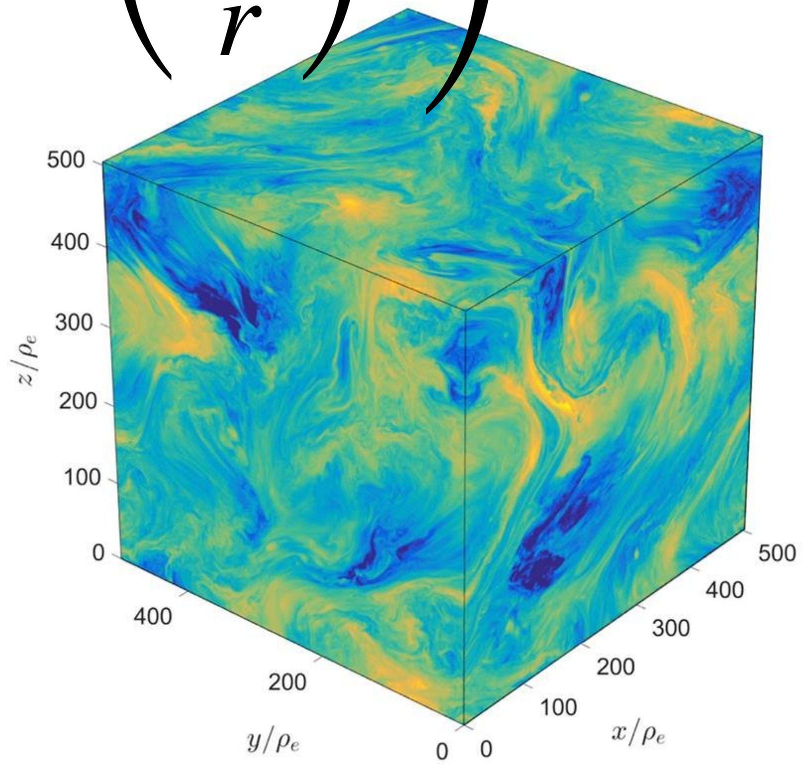
- Maldacena, 1997
- Witten, 1998
- Bhattacharyya et. al. 2007

$$ds^2 = r^2(-f(r)dt^2 + (dx^1)^2 + (dx^2)^2) + \frac{dr^2}{r^2 f(r)}$$

$$f(r) = \left(1 - \left(\frac{r_0}{r}\right)^3\right)$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = (\epsilon + P)\, u^\mu u^\nu + P\eta^{\mu\nu} + \ldots$$



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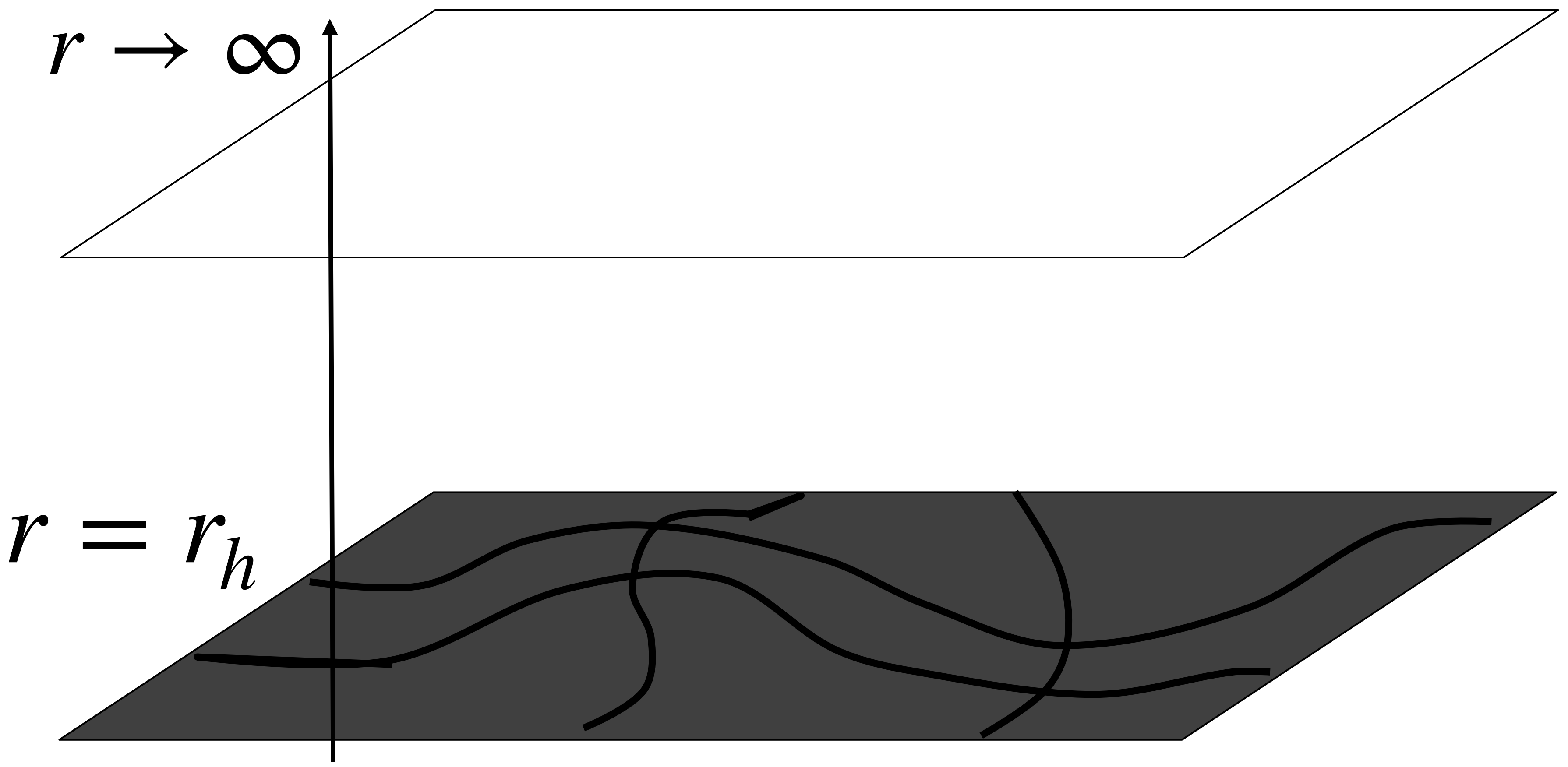

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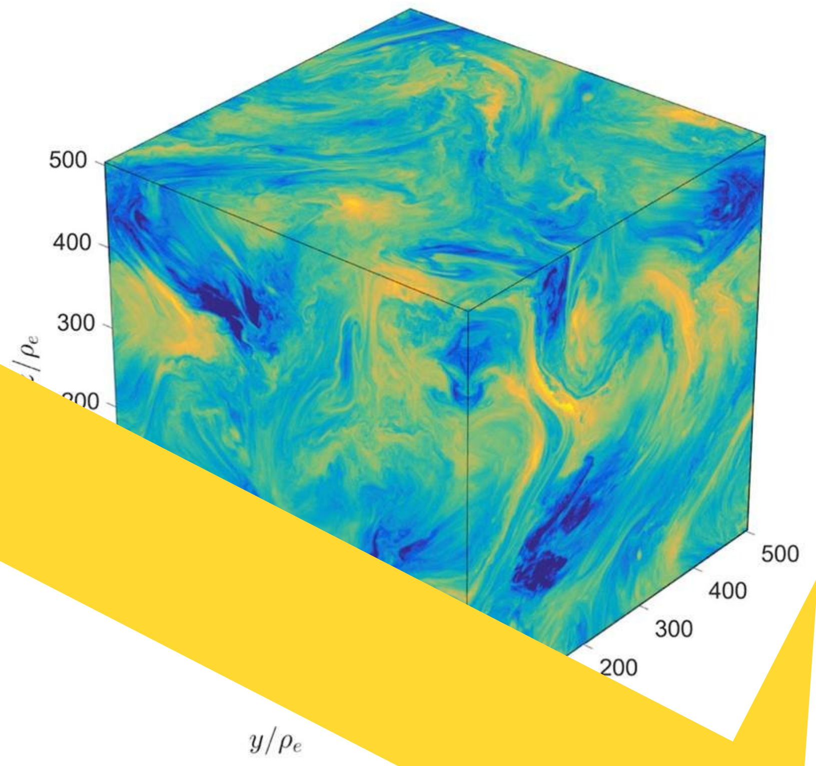

$$\frac{v}{c} \ll 1$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = (\epsilon + P)\, u^\mu u^\nu + P\eta^{\mu\nu} + \dots$$

$$\dot{\vec{v}} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p + \nu \nabla^2 \vec{v}$$

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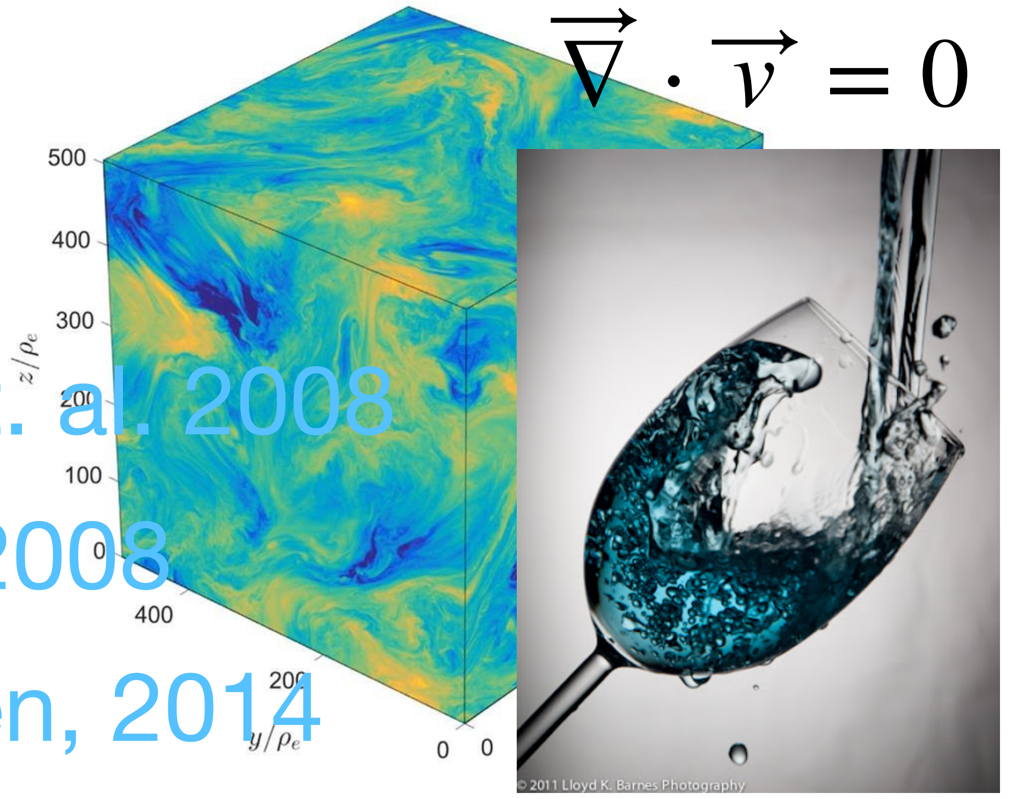

- Bhattacharyya et. al. 2008
- Fouxon and Oz 2008
- Karch and Jensen, 2014

Holographic turbulence

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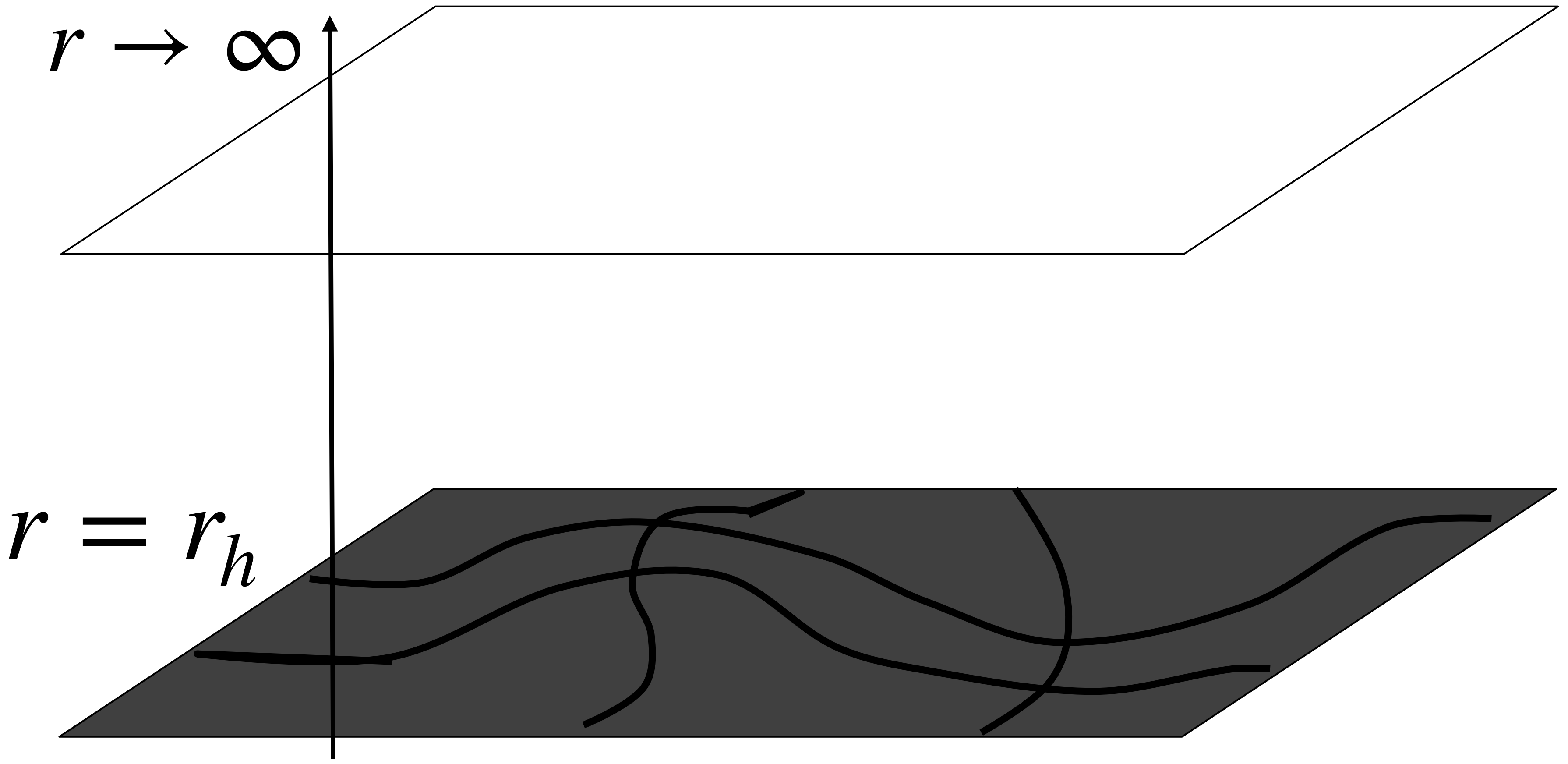

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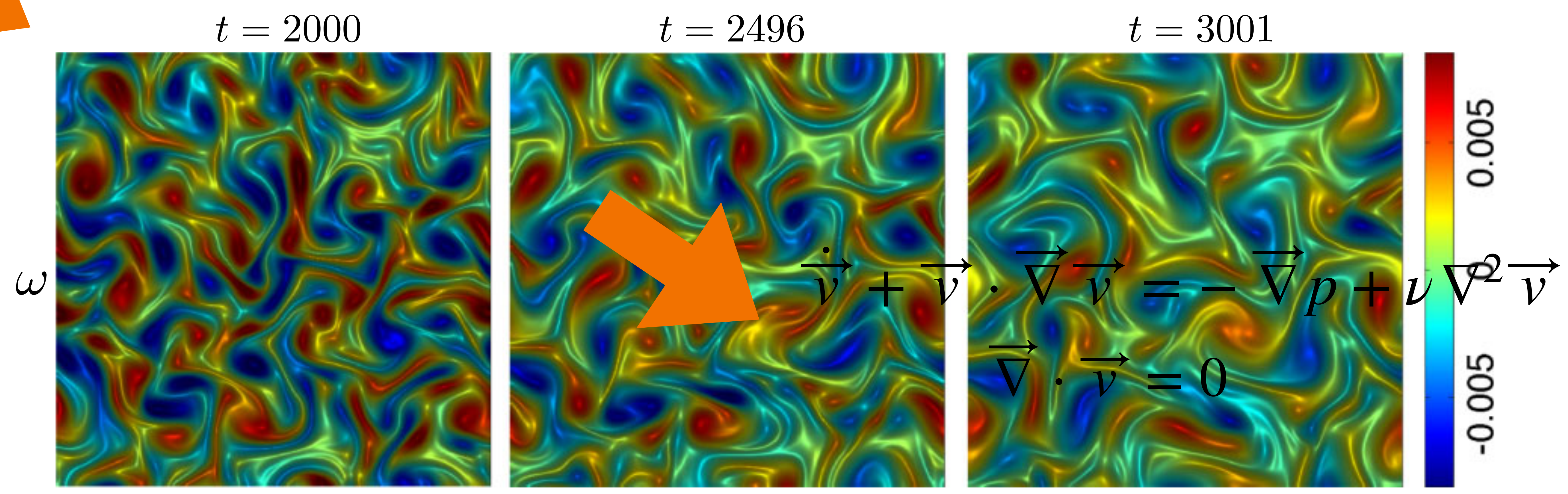
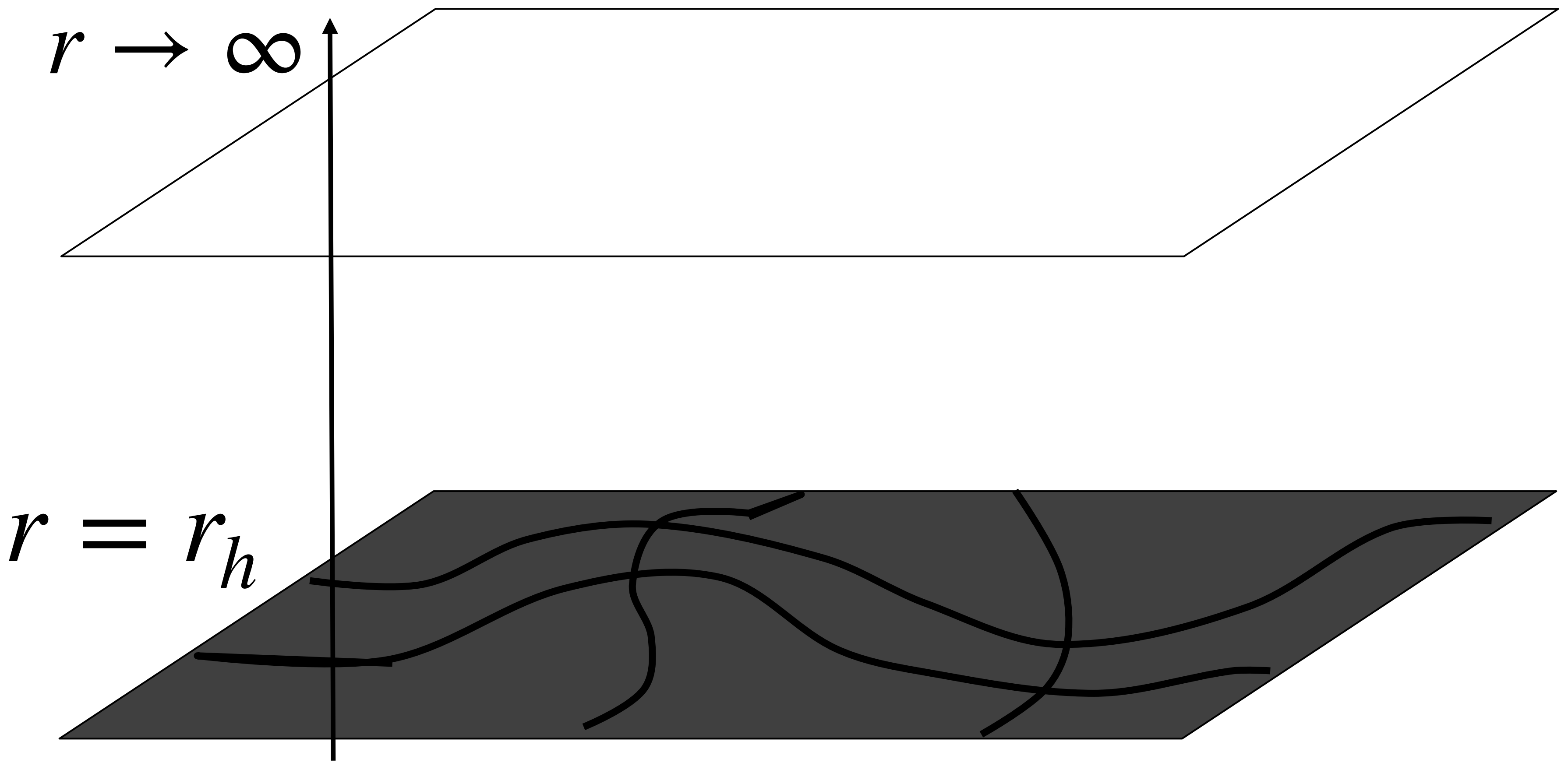


Holographic turbulence

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$



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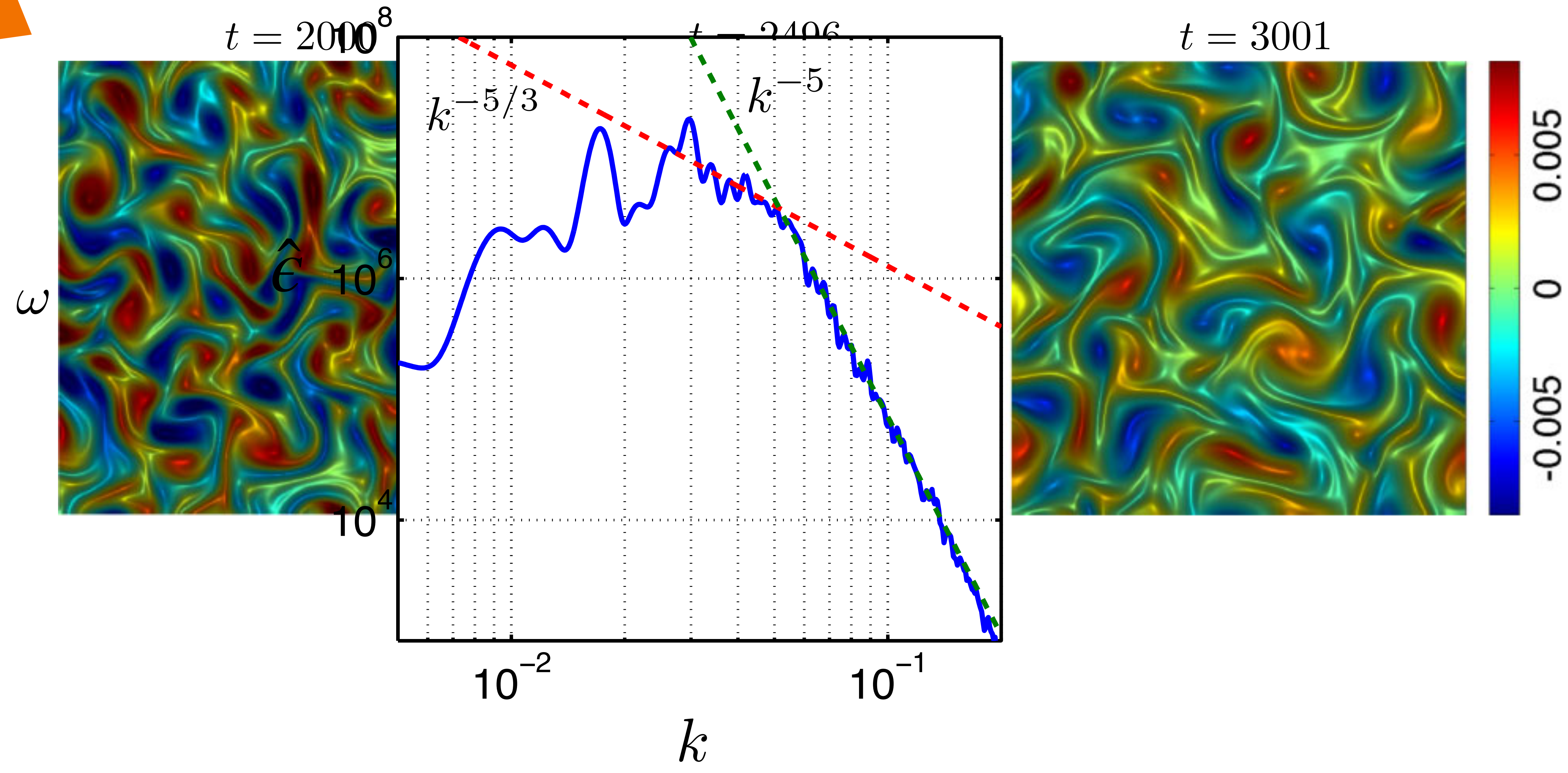

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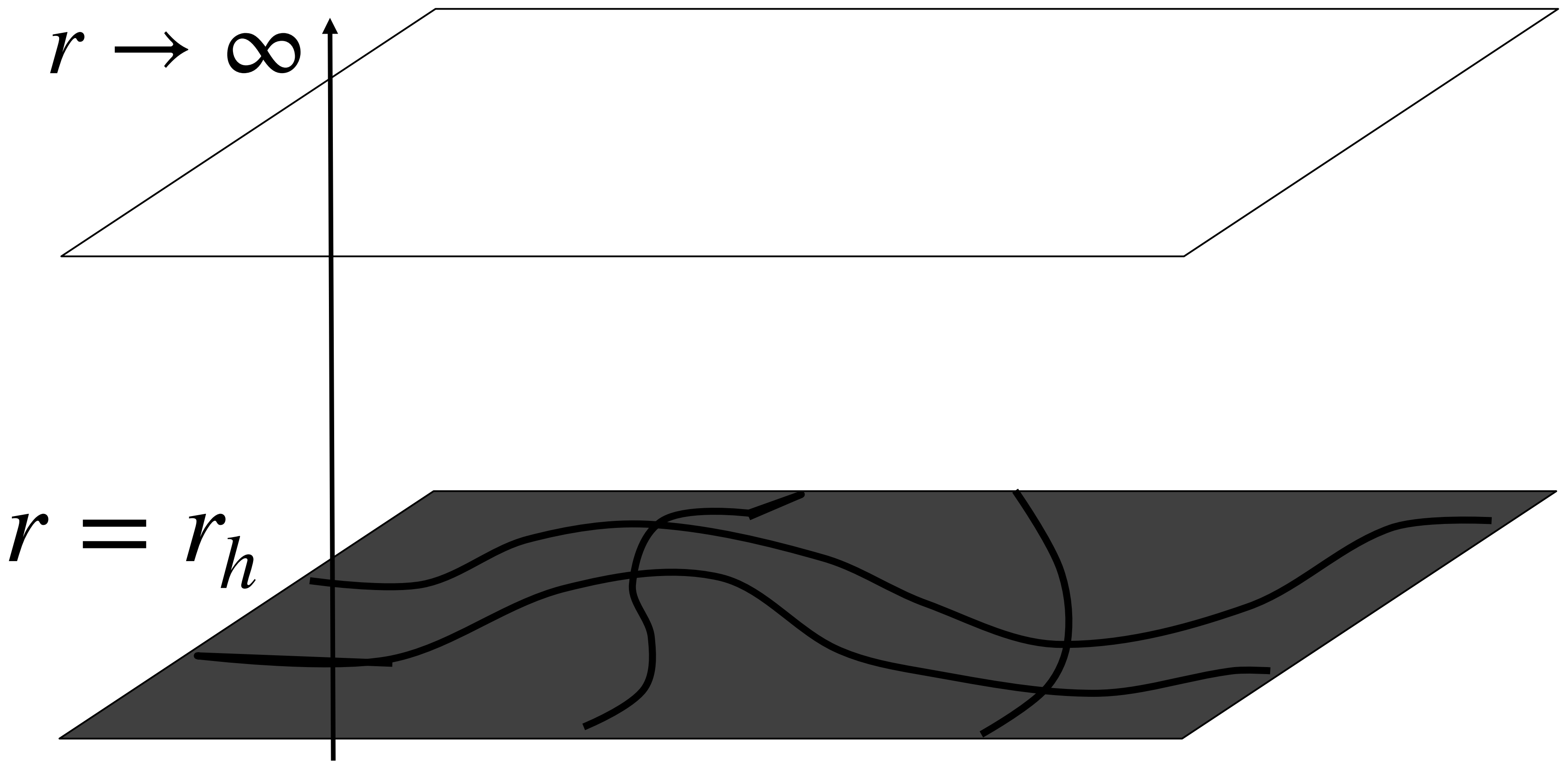

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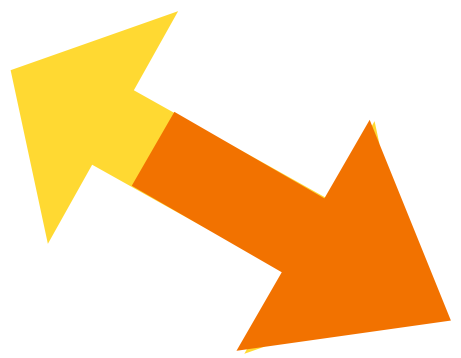

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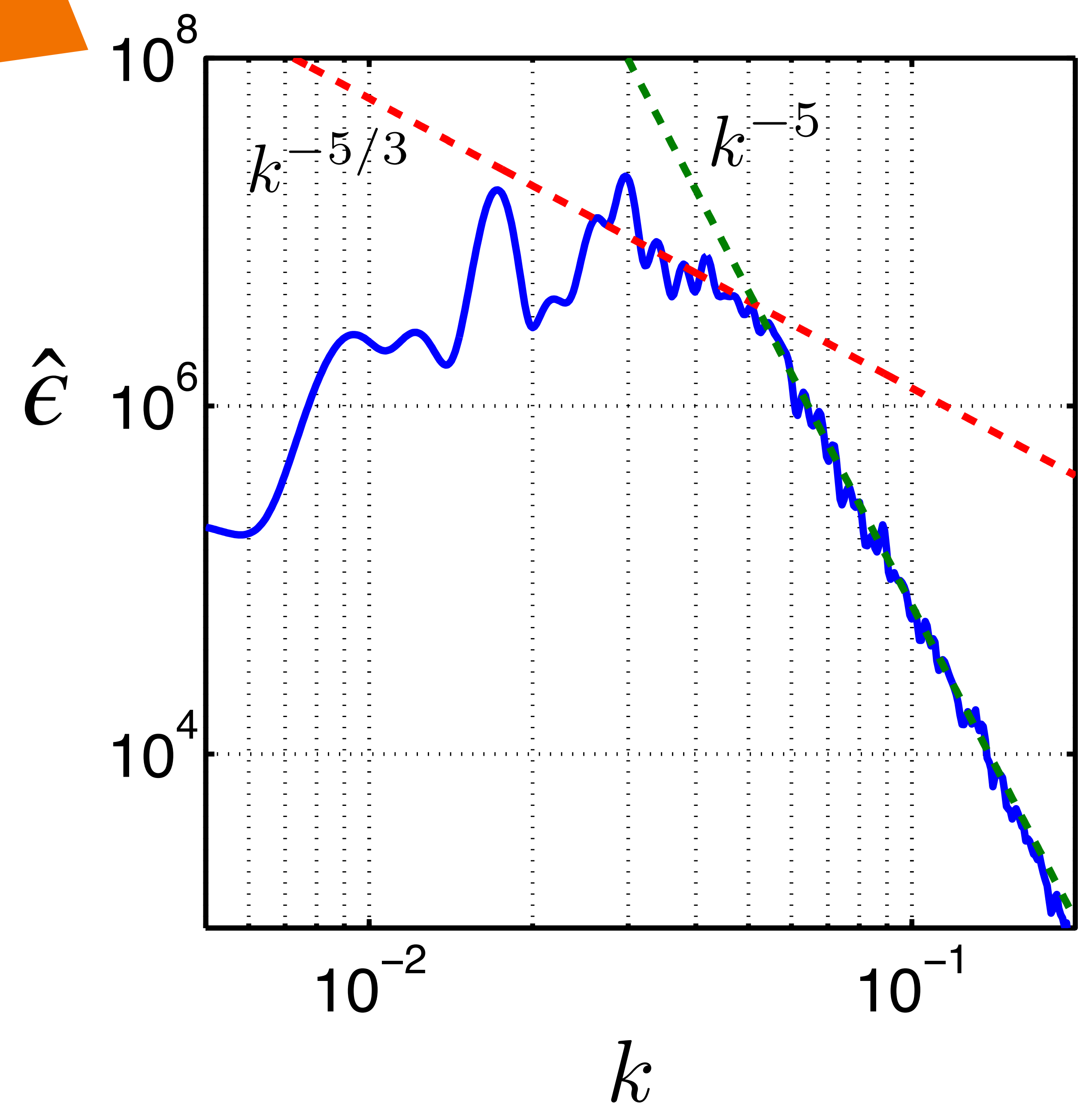
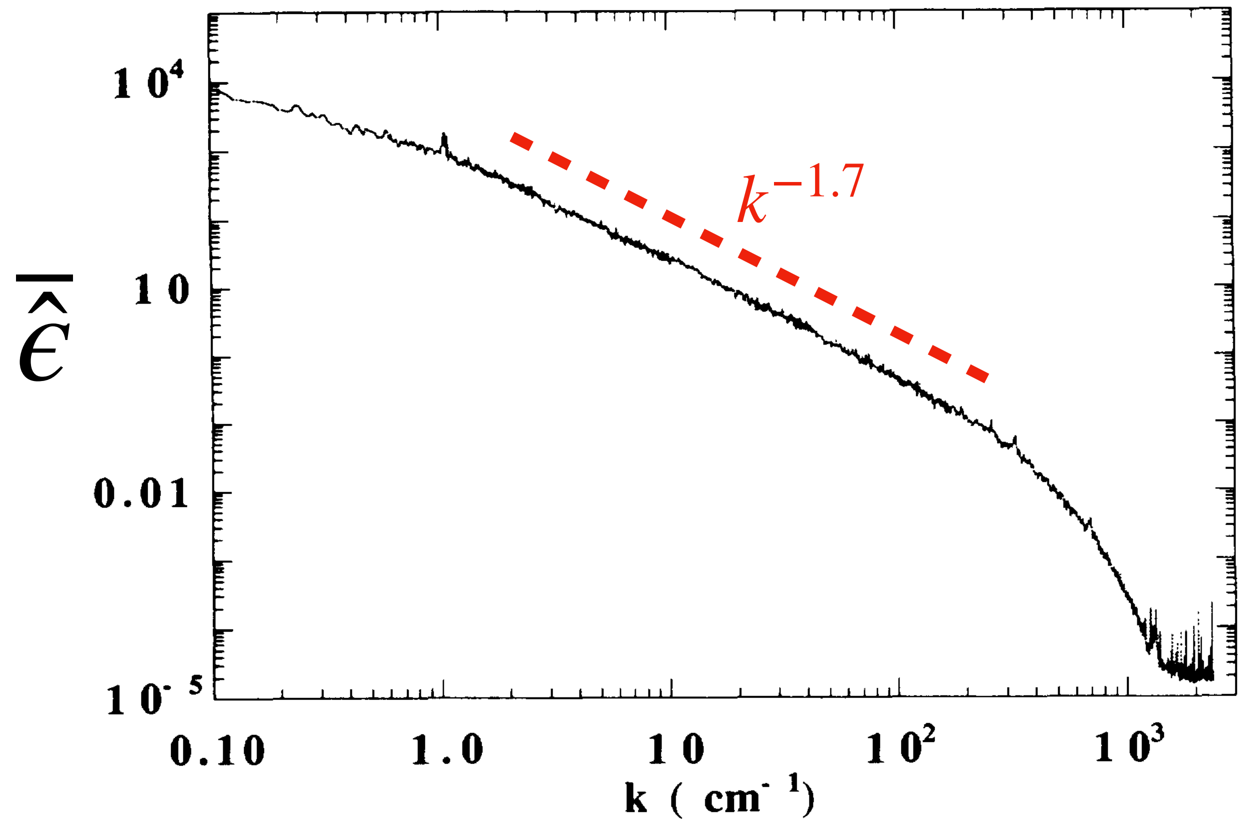


Recall:



Holographic turbulence

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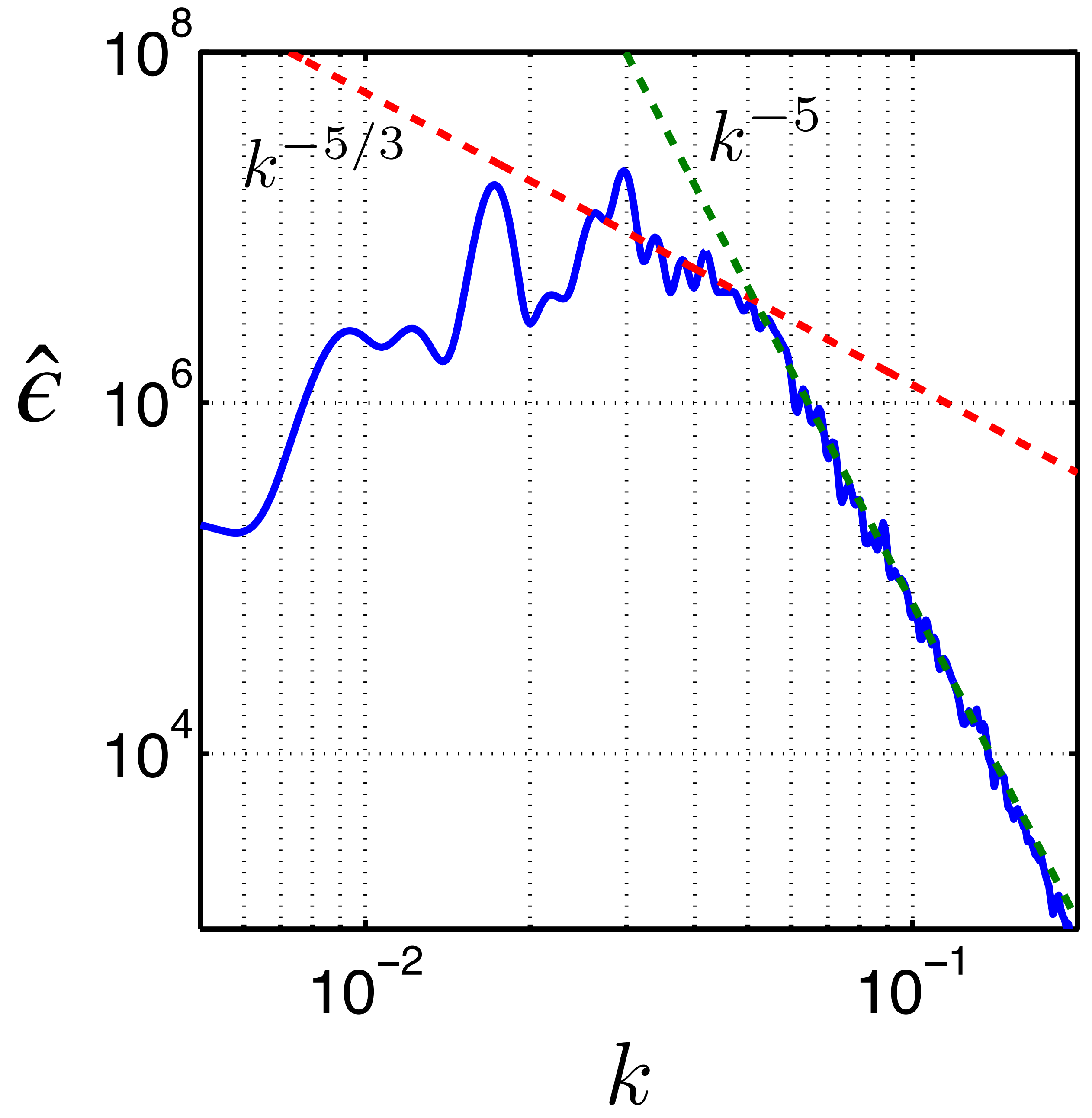

vrs.

$$\dot{\vec{v}} + \vec{v} \cdot \vec{\nabla} \vec{v} = - \vec{\nabla} p + \nu \nabla \quad \vec{f}$$
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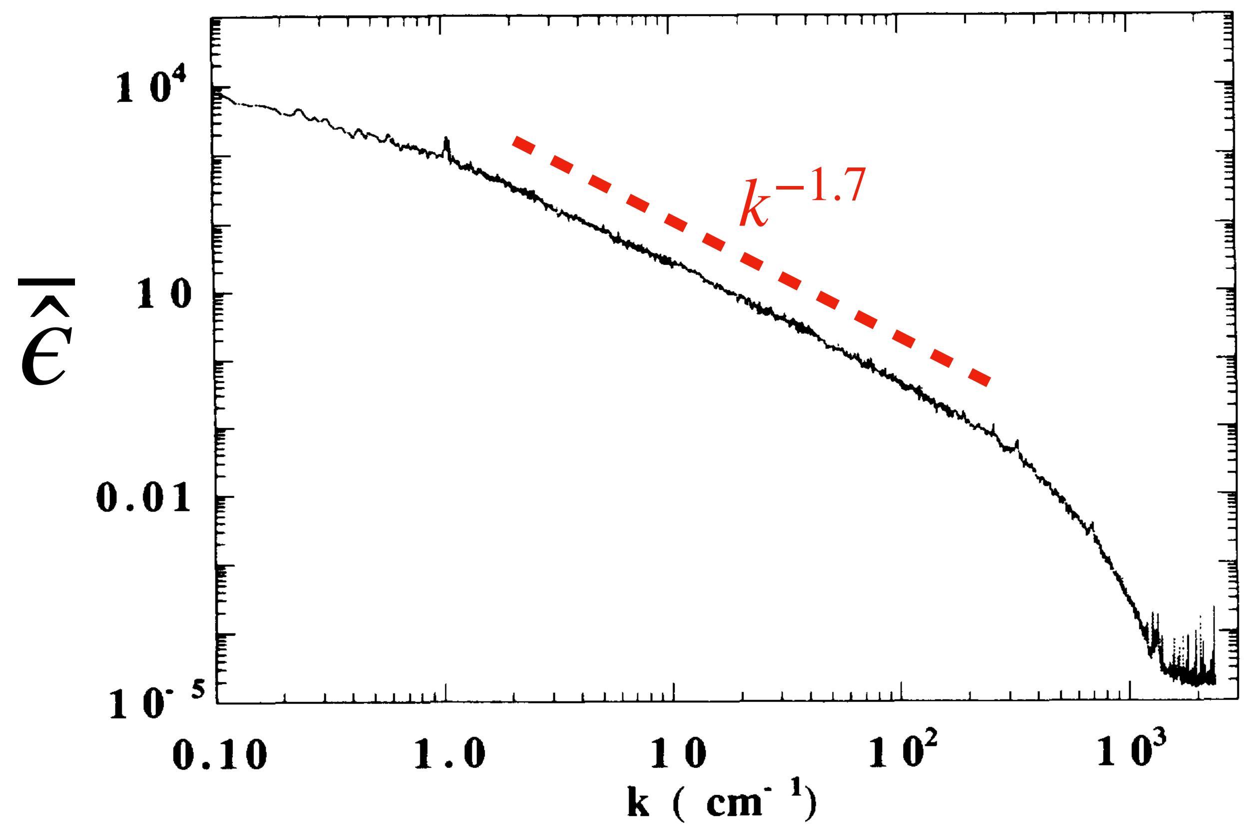

Holographic turbulence

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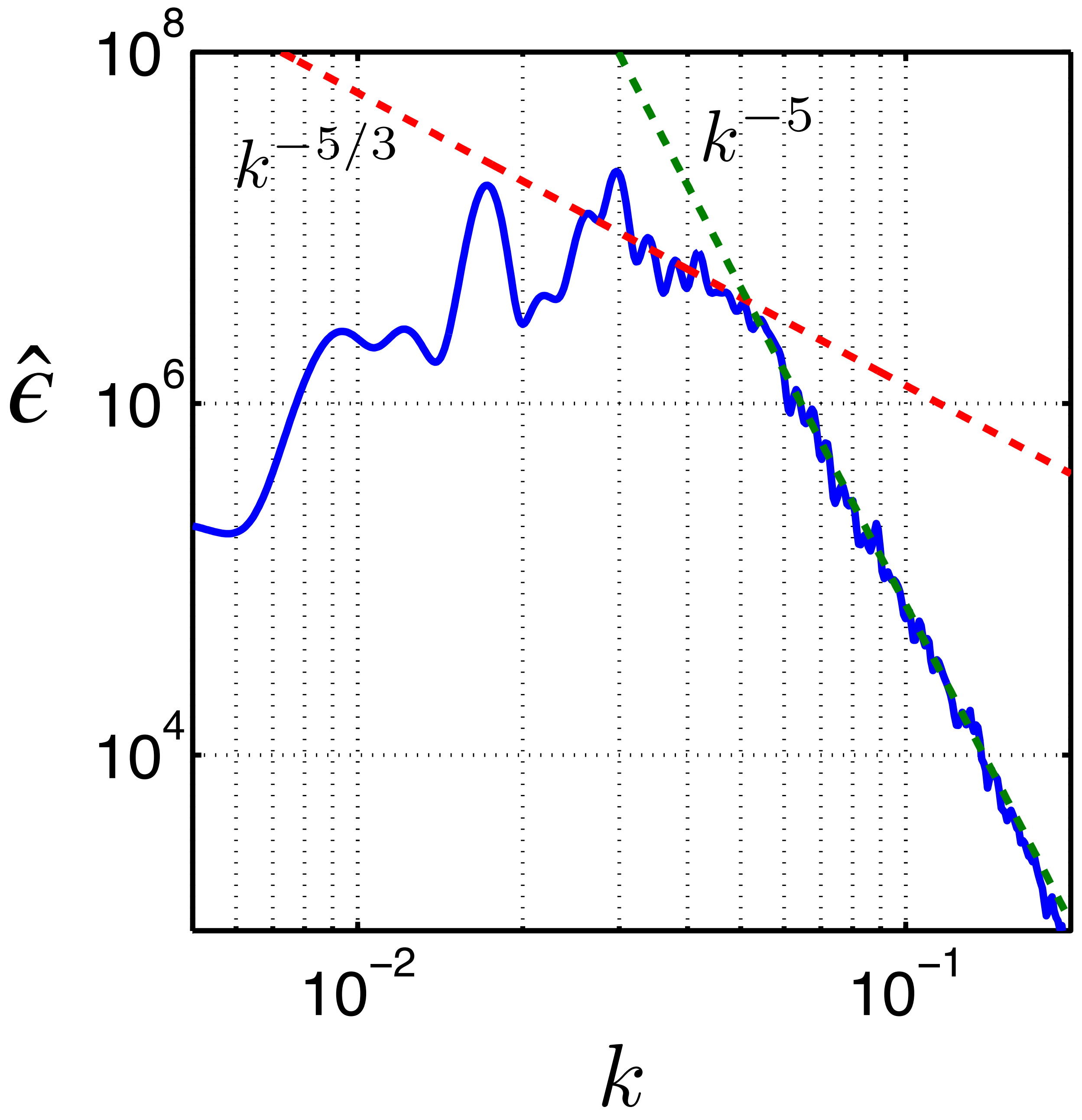

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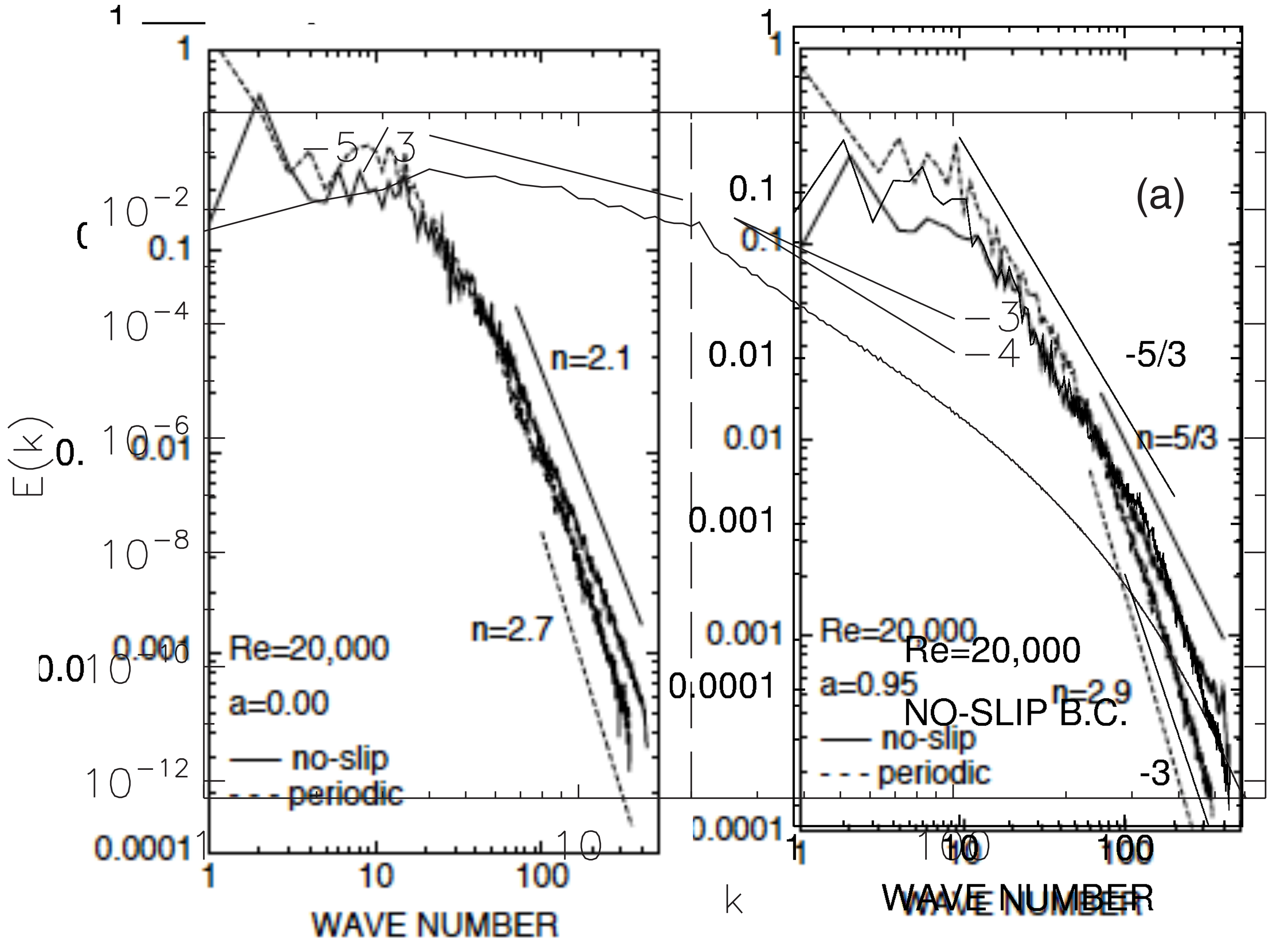

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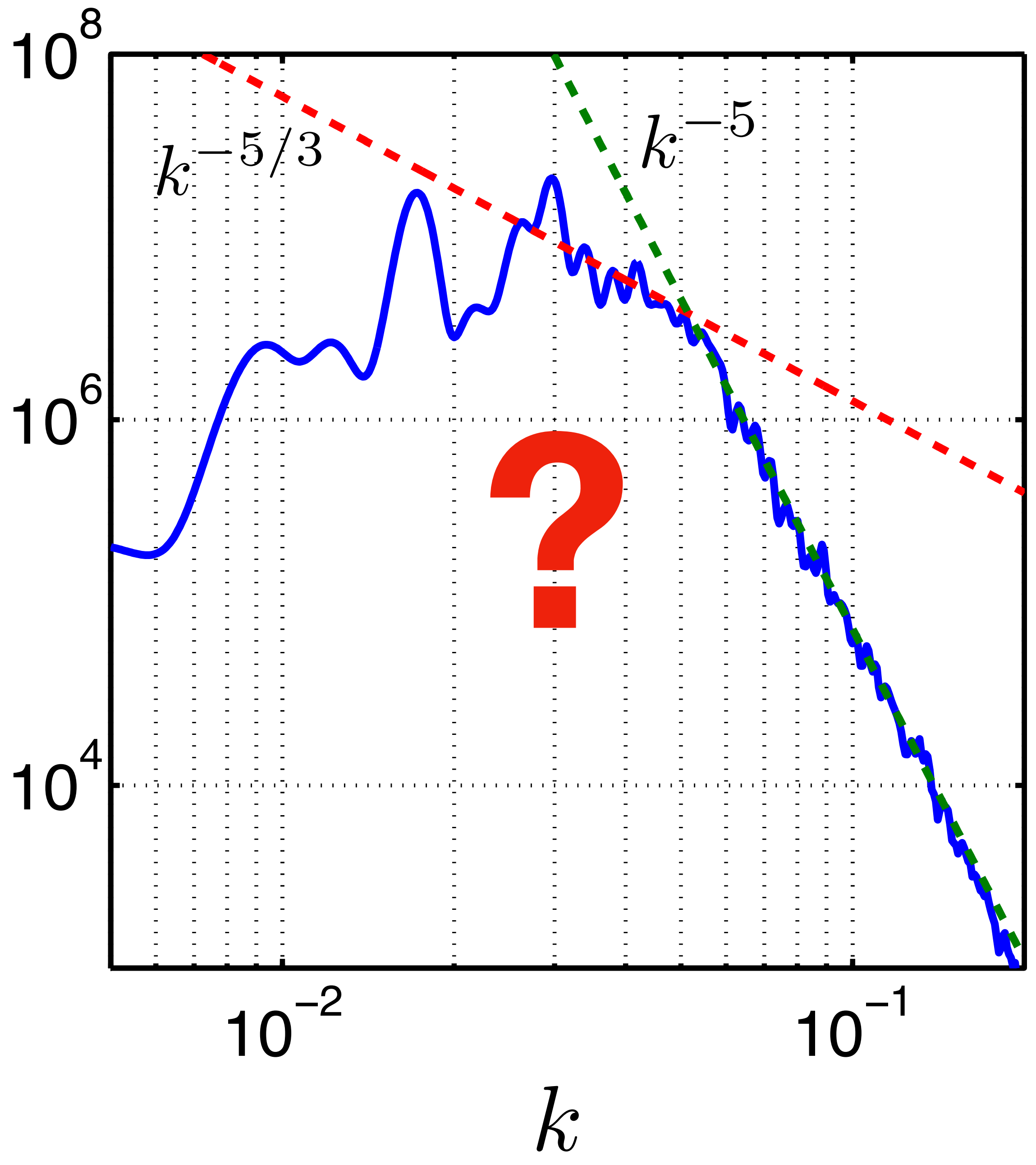
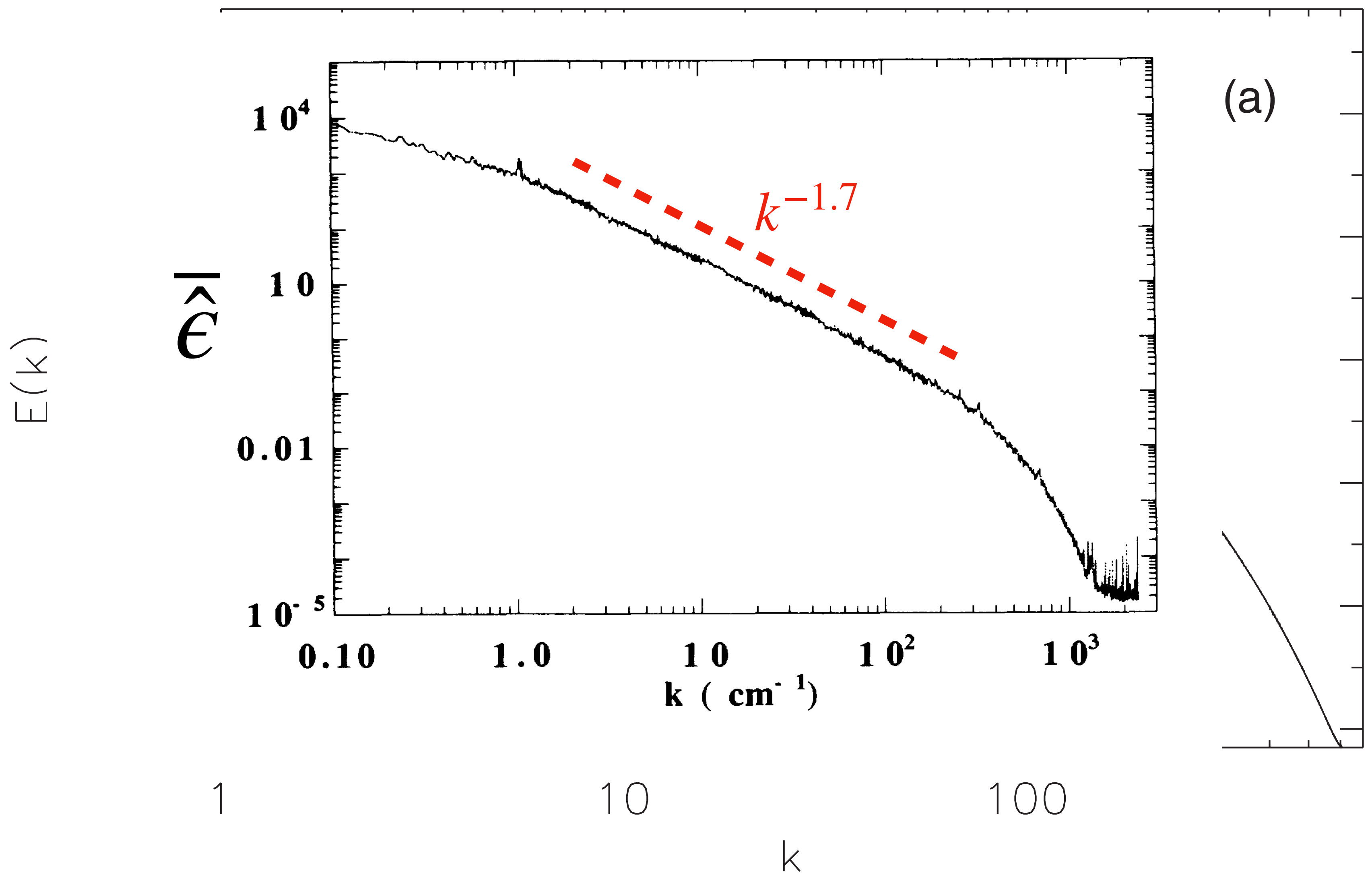
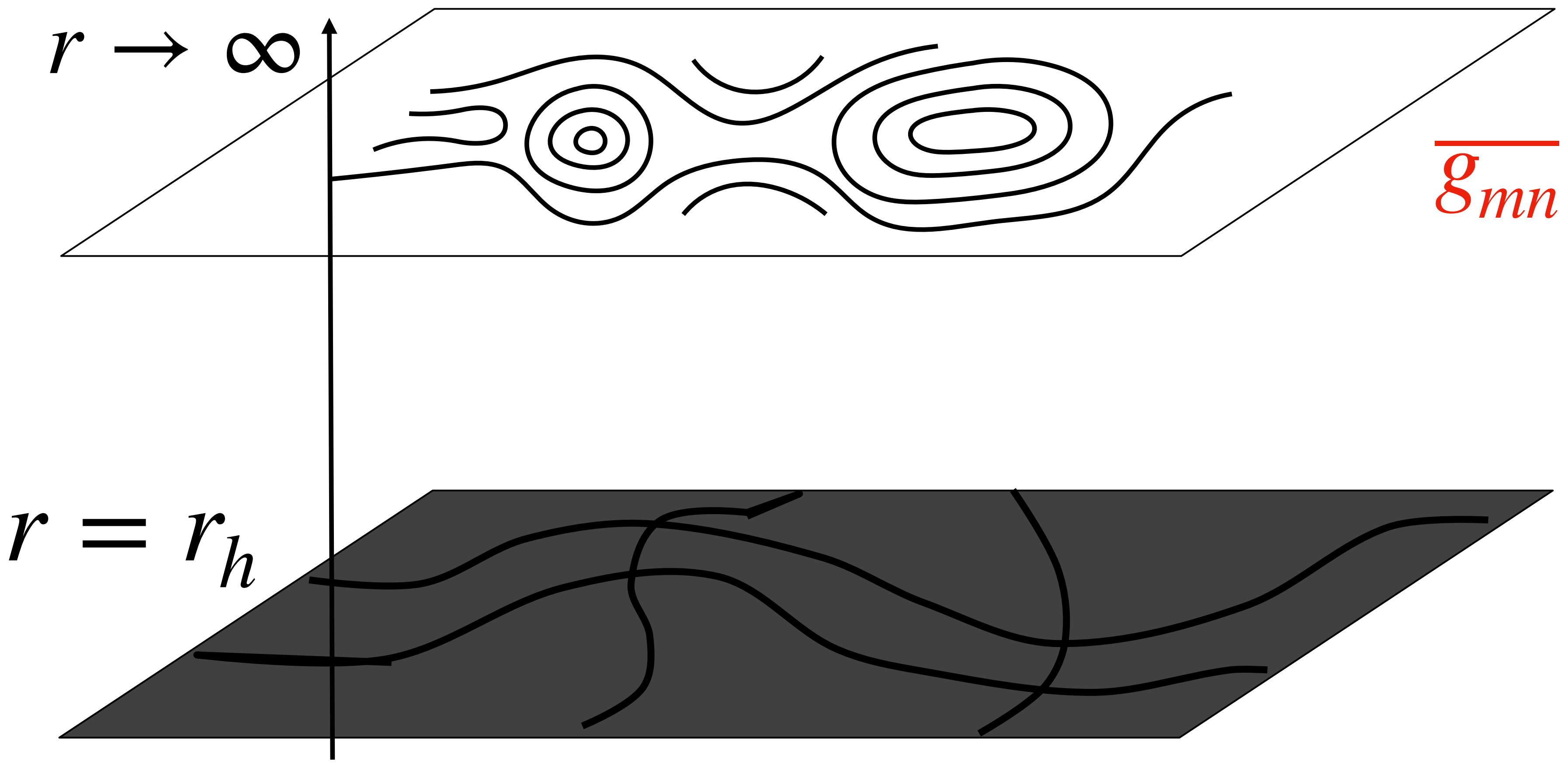


vrs.



Holographic duality
Stochastic gravity and turbulence

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$



$$\overline{g_{ij}} = \delta_{ij}$$

$$\dot{\vec{v}} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p + \nu \nabla^2 \vec{v} + \vec{f}$$

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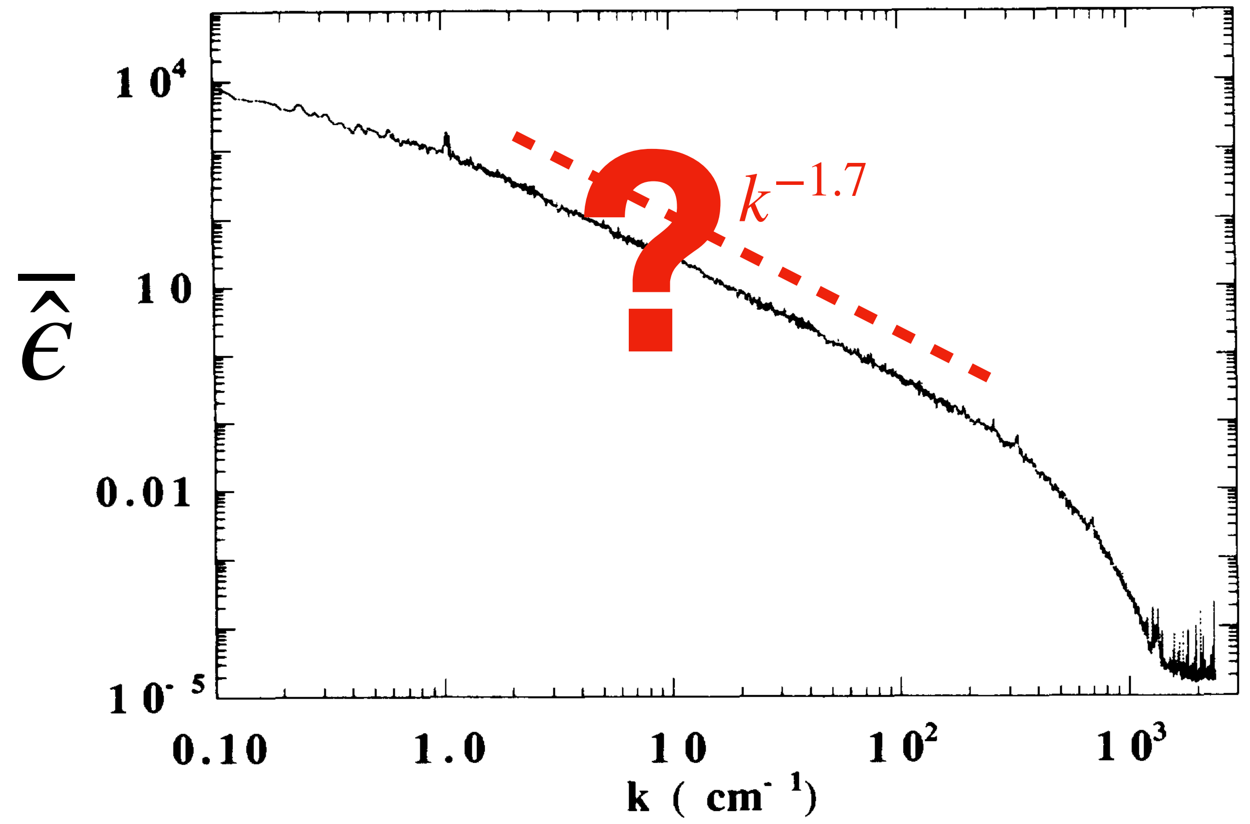
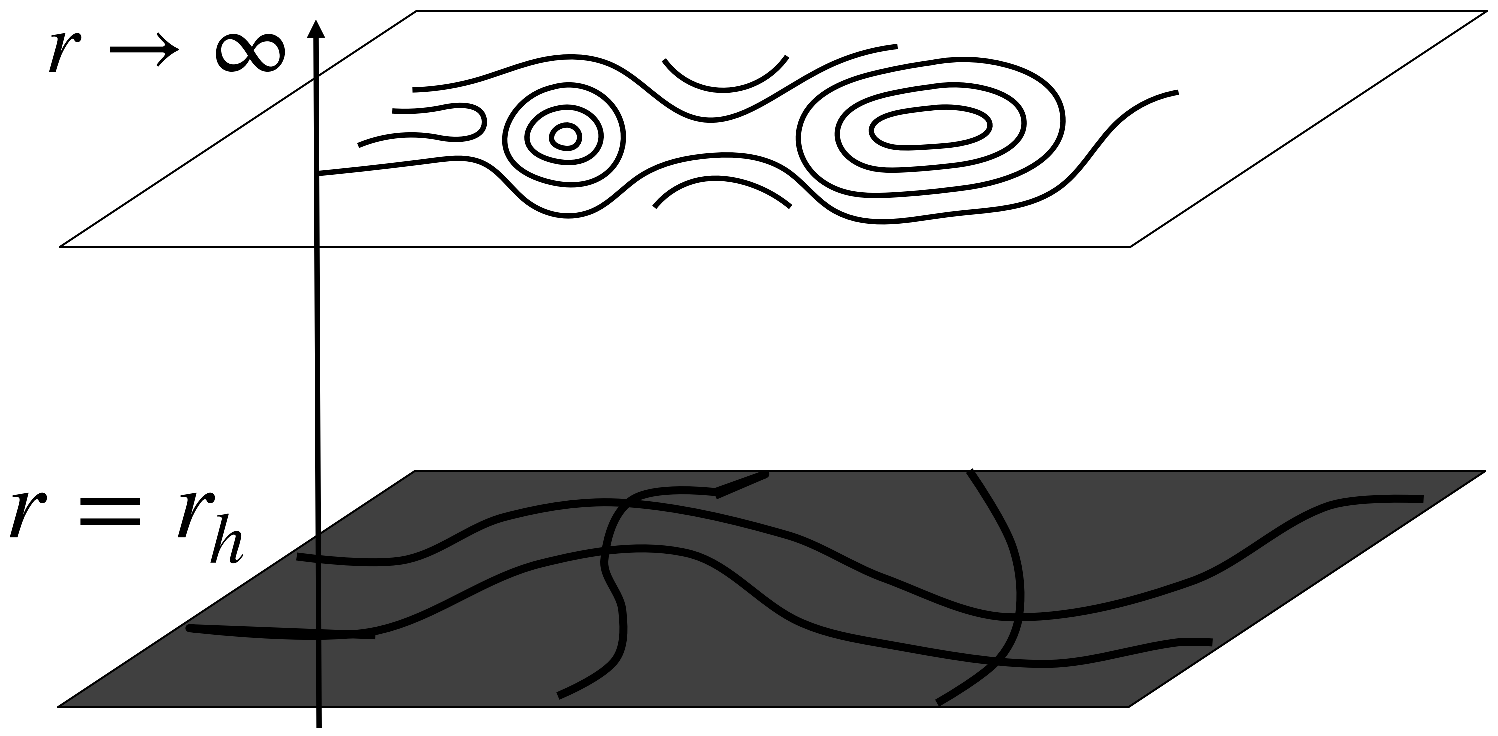
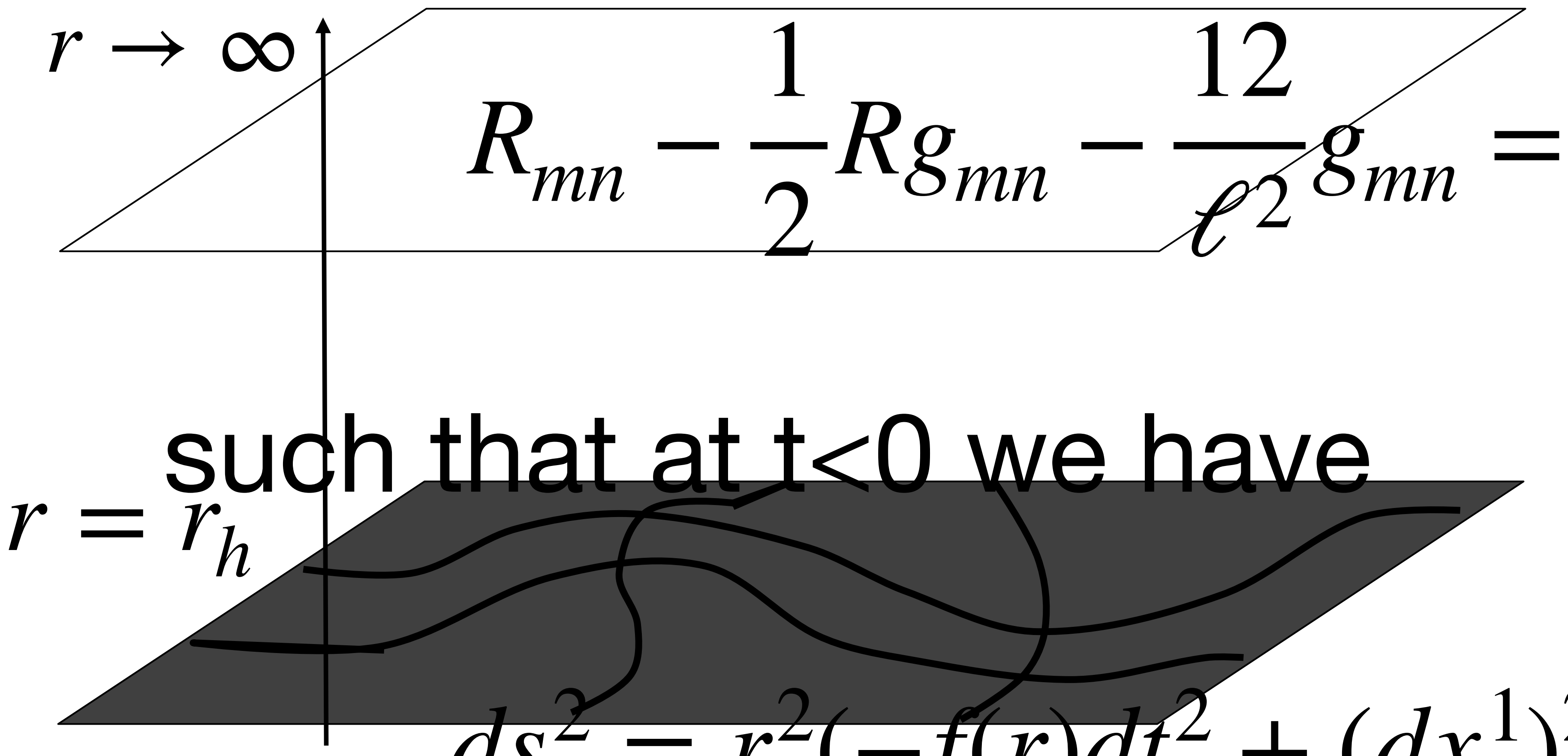


Stochastic gravity and turbulence

We wish to solve

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$





$$R_{mn} - \frac{1}{2} R g_{mn} - \frac{12}{\ell^2} g_{mn} = 0$$

such that at t<0 we have

$$ds^2 = r^2(-f(r)dt^2 + (dx^1)^2 + (dx^2)^2) + \frac{dr^2}{r^2 f(r)}$$

$$f(r) = \left(1 - \left(\frac{r_0}{r}\right)^3\right)$$

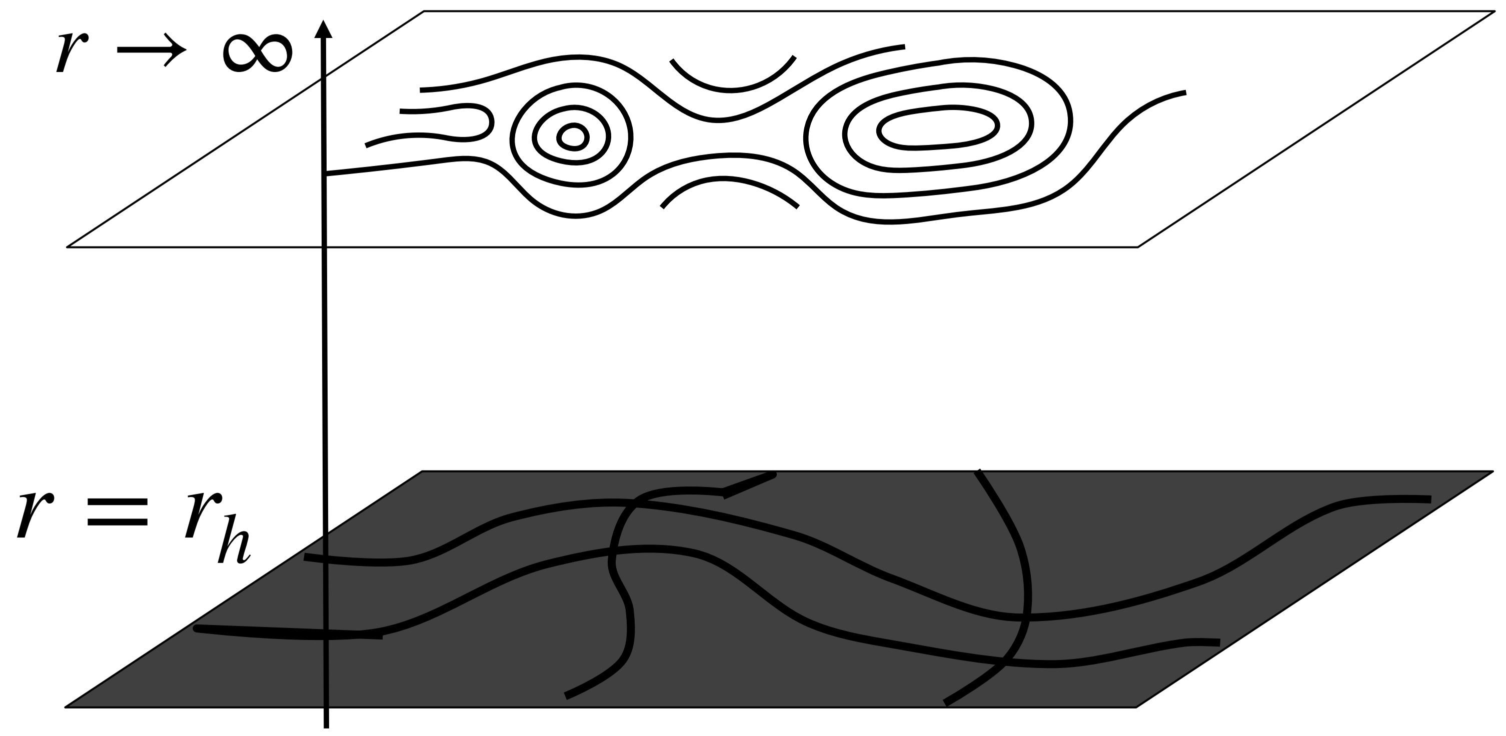
At t>0 we have

$$g_{mn} \xrightarrow[r\to\infty]{} g^{(0)}_{\mu\nu}$$

$$g^{(0)}_{\mu\nu} dx^\mu dx^\nu = -(1+Q)dt^2 + (dx^1)^2 + (dx^2)^2$$

Stochastic gravity and turbulence

We wish to solve



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At t>0 we have

$$g_{mn} \xrightarrow[r\to\infty]{} g^{(0)}_{\mu\nu} \qquad g^{(0)}_{\mu\nu} dx^\mu dx^\nu = -(1+Q)dt^2 + (dx^1)^2 + (dx^2)^2$$

where

$$Q = q \qquad \dot{q} = -\frac{q}{\tau} + \frac{\xi}{\tau}$$

$$\overline{\xi(t, \overrightarrow{x})} = 0 \qquad \overline{\xi(t, \overrightarrow{x})\xi(t', \overrightarrow{x}')} = D(\overrightarrow{x} - \overrightarrow{x}')\delta(t - t') \qquad \hat{D}(\overrightarrow{k}) = \delta(|\overrightarrow{k}| - k_f)$$

Stochastic gravity and turbulence

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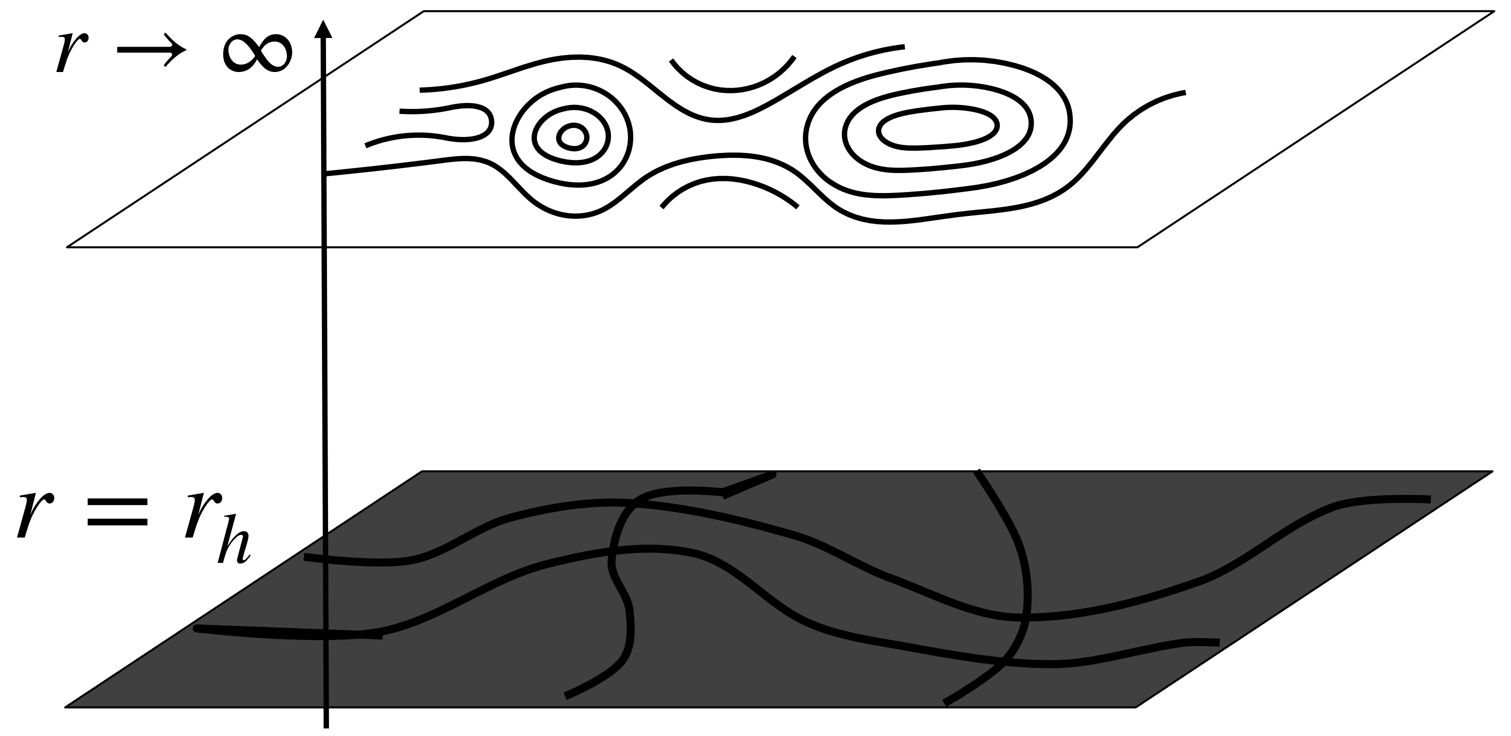

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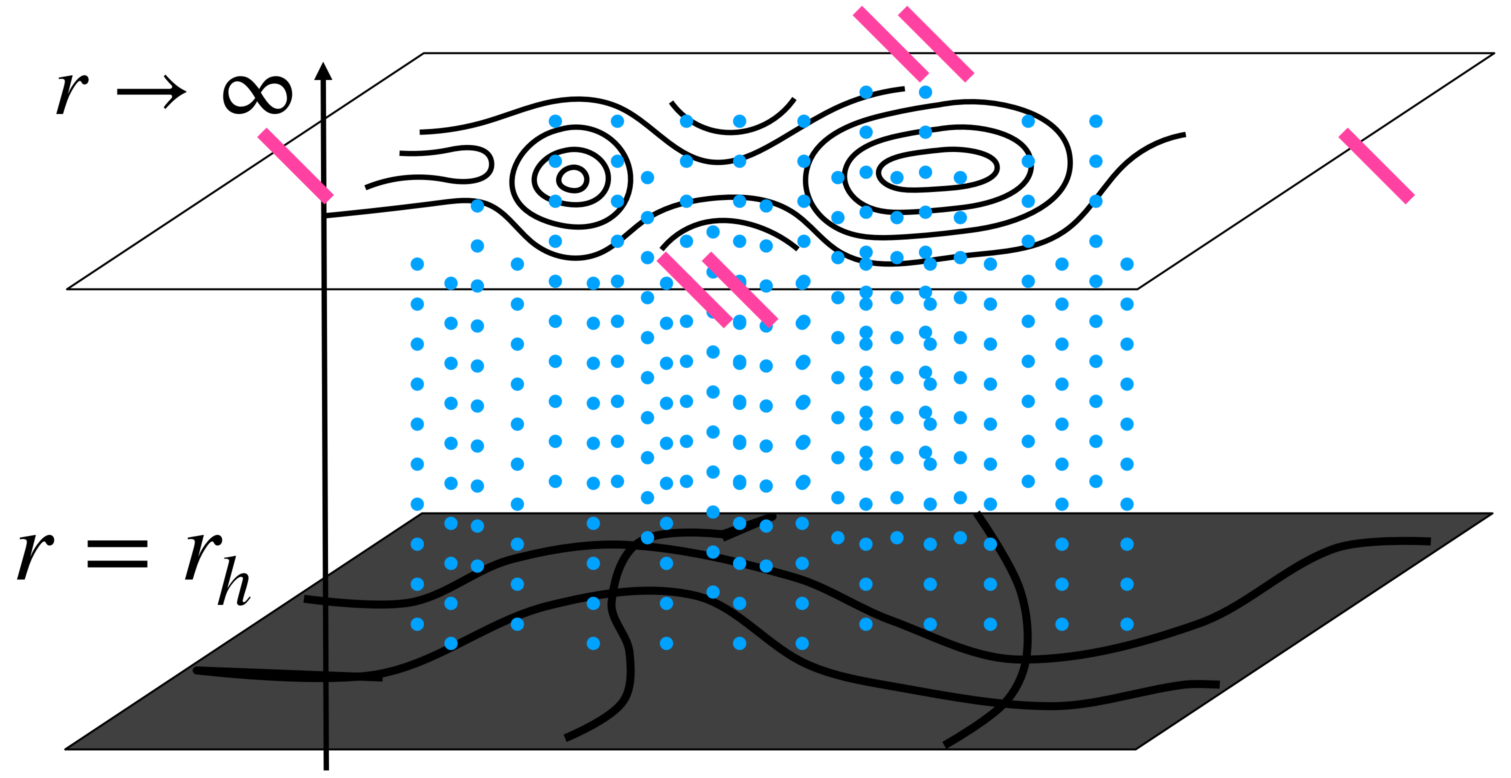
$$g_{mn} \xrightarrow[r\to\infty]{} g^{(0)}_{\mu\nu} \qquad g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -(1+Q)dt^2 + (dx^1)^2 + (dx^2)^2$$

The energy momentum tensor, $T^{\mu\nu}$, can be read off of the metric.

After averaging we obtain $\overline{T^{\mu\nu}}$.

Stochastic gravity and turbulence

We wish to solve



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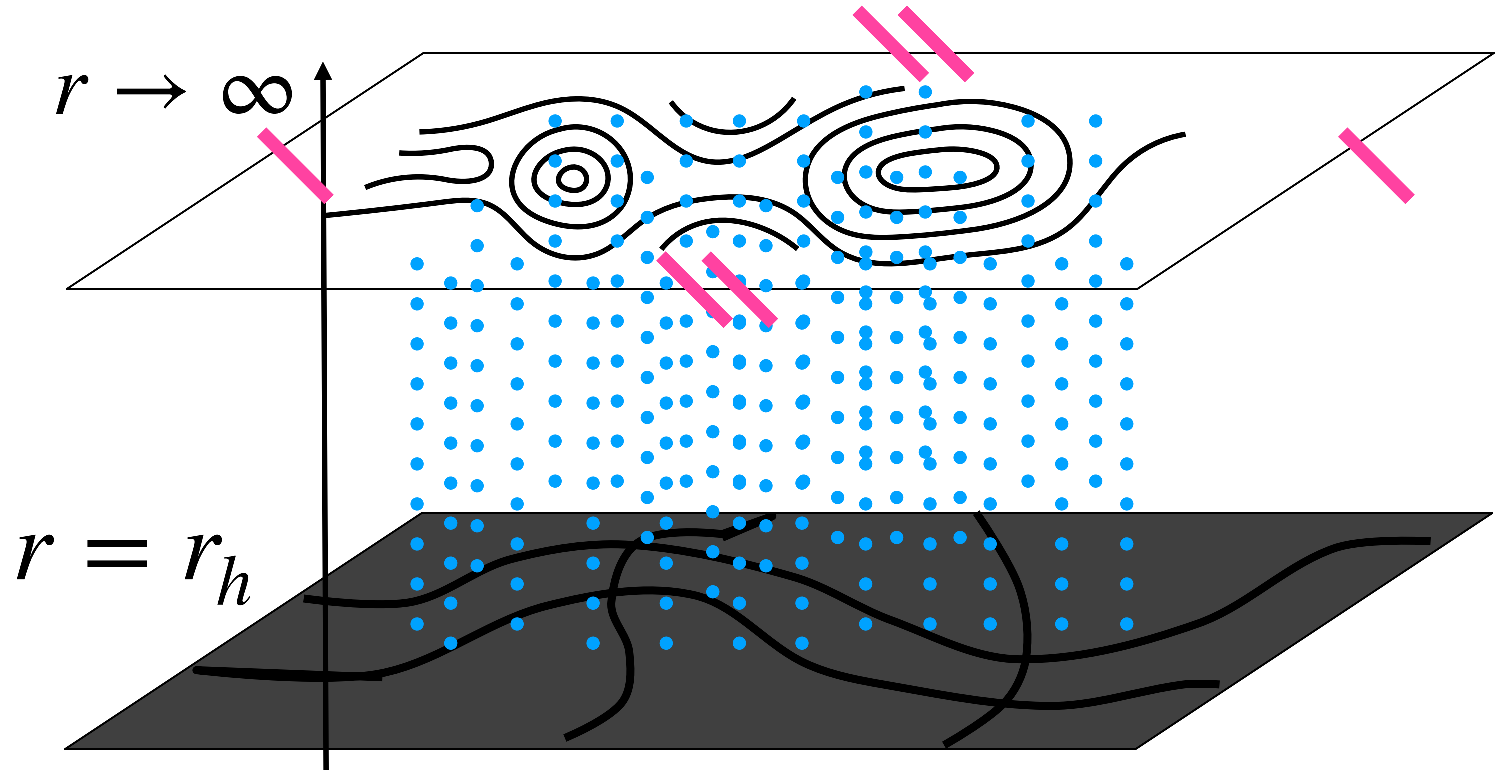
In practice, we need to solve numerically.

Solving in the right order allows us to rewrite the Einstein equations as a set of ordinary stochastic differential equations. (Chesler, Yaffe, 2013)

Stochastic gravity and turbulence

We solved



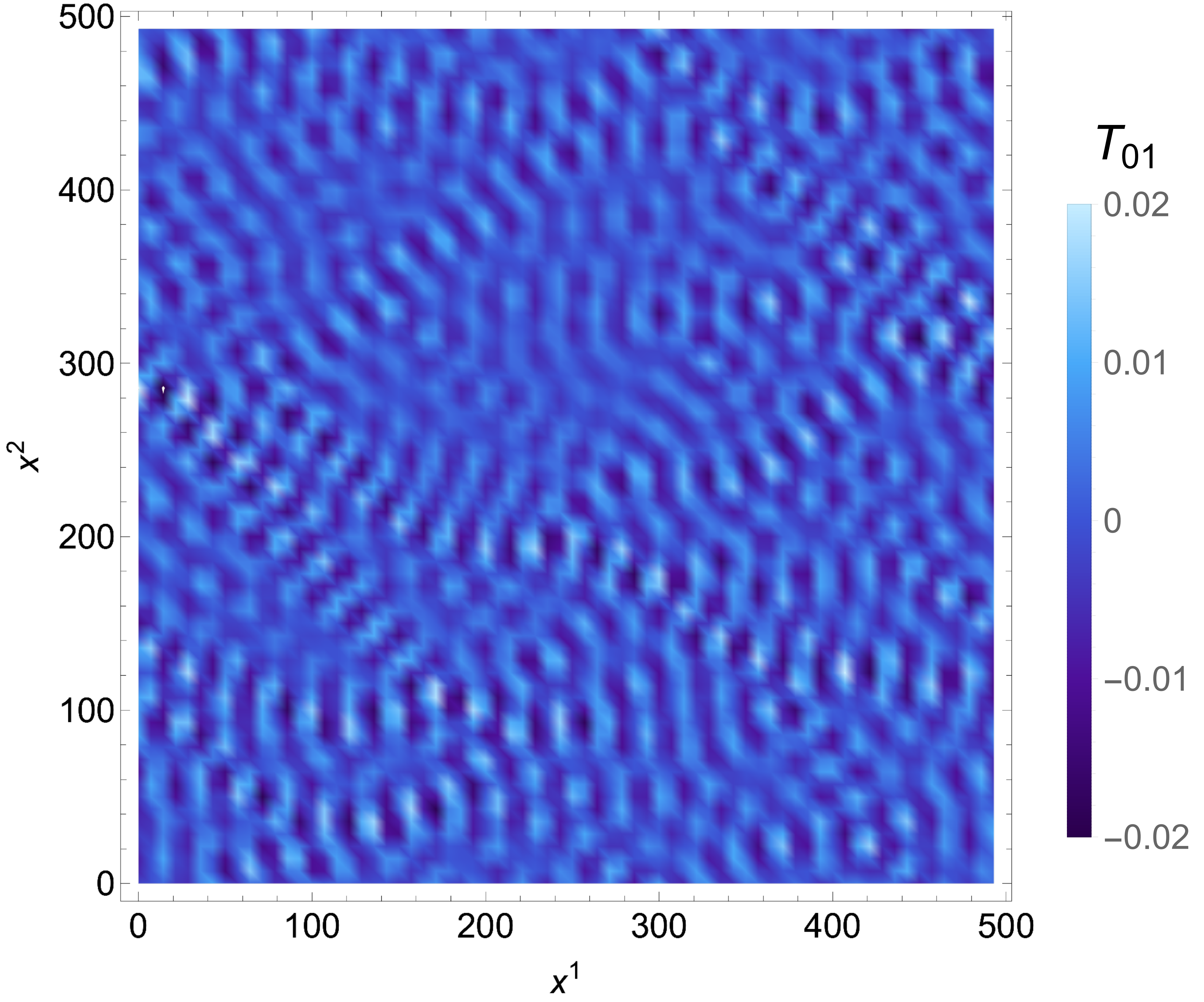
$$R_{mn} - \frac{1}{2} R g_{mn} - \frac{12}{\ell^2} g_{mn} = 0$$

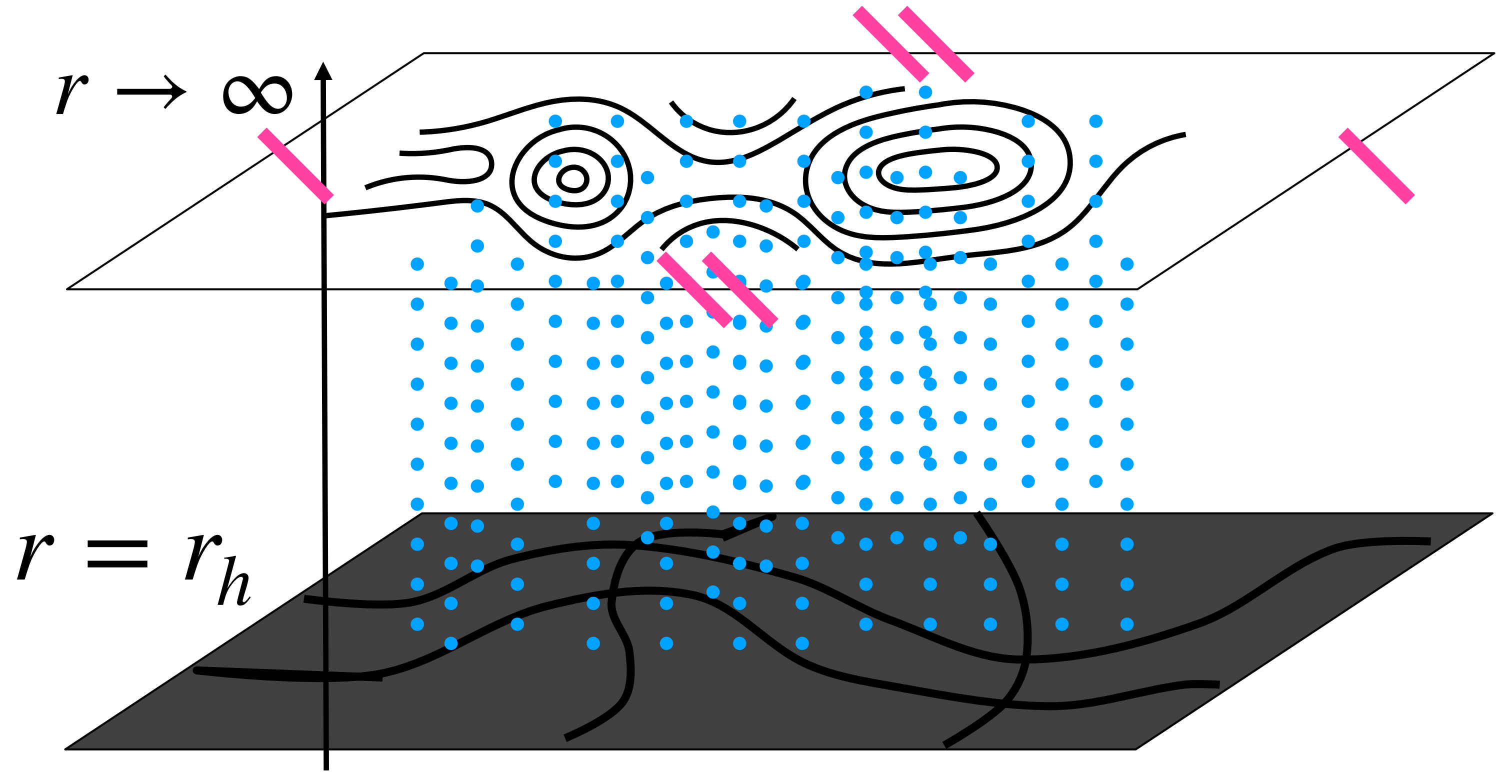
Did this many times, and then computed the average:

$$\overline{T_{\mu\nu}}$$

Stochastic gravity and turbulence

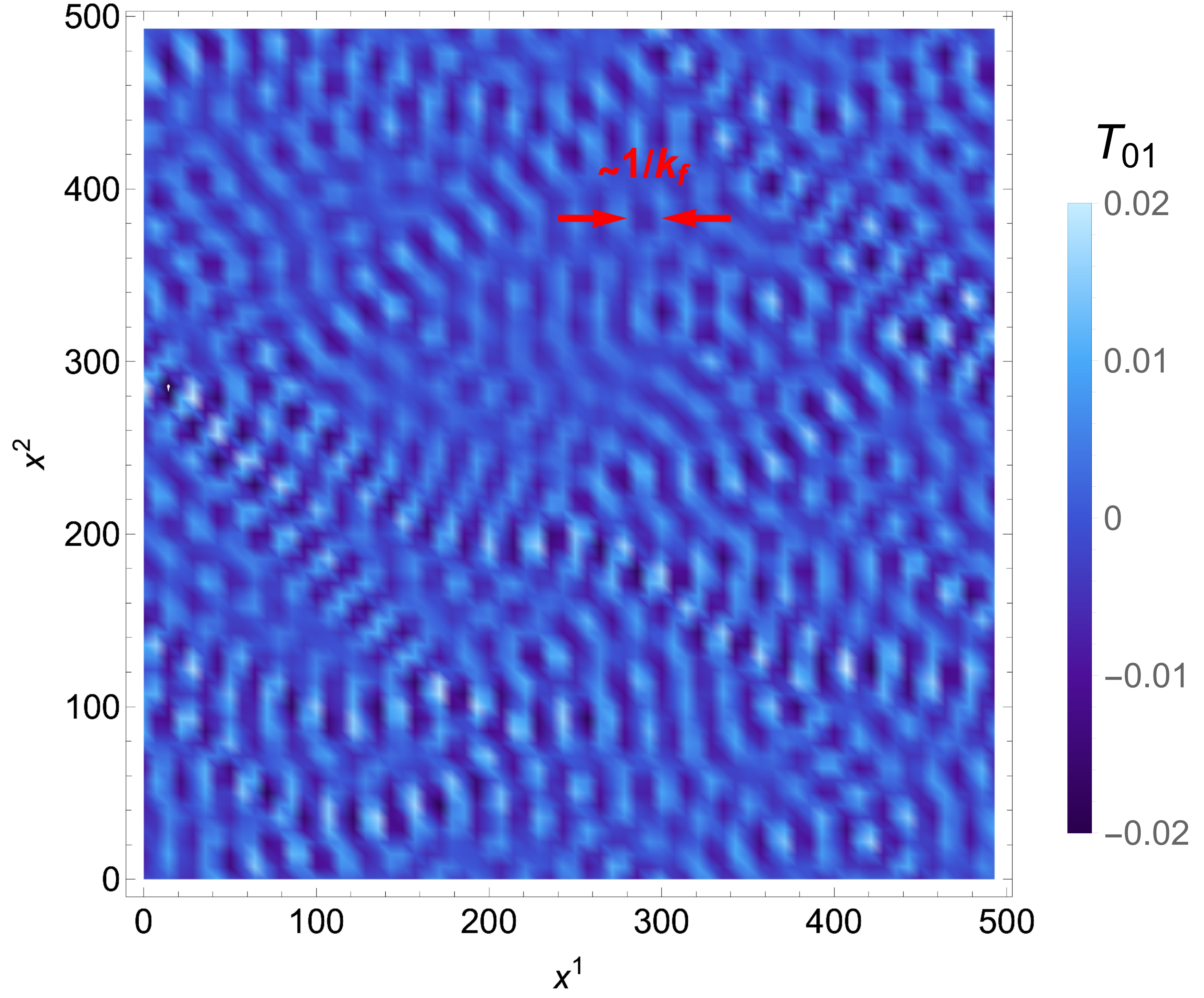
$\overline{T_{\mu\nu}}$

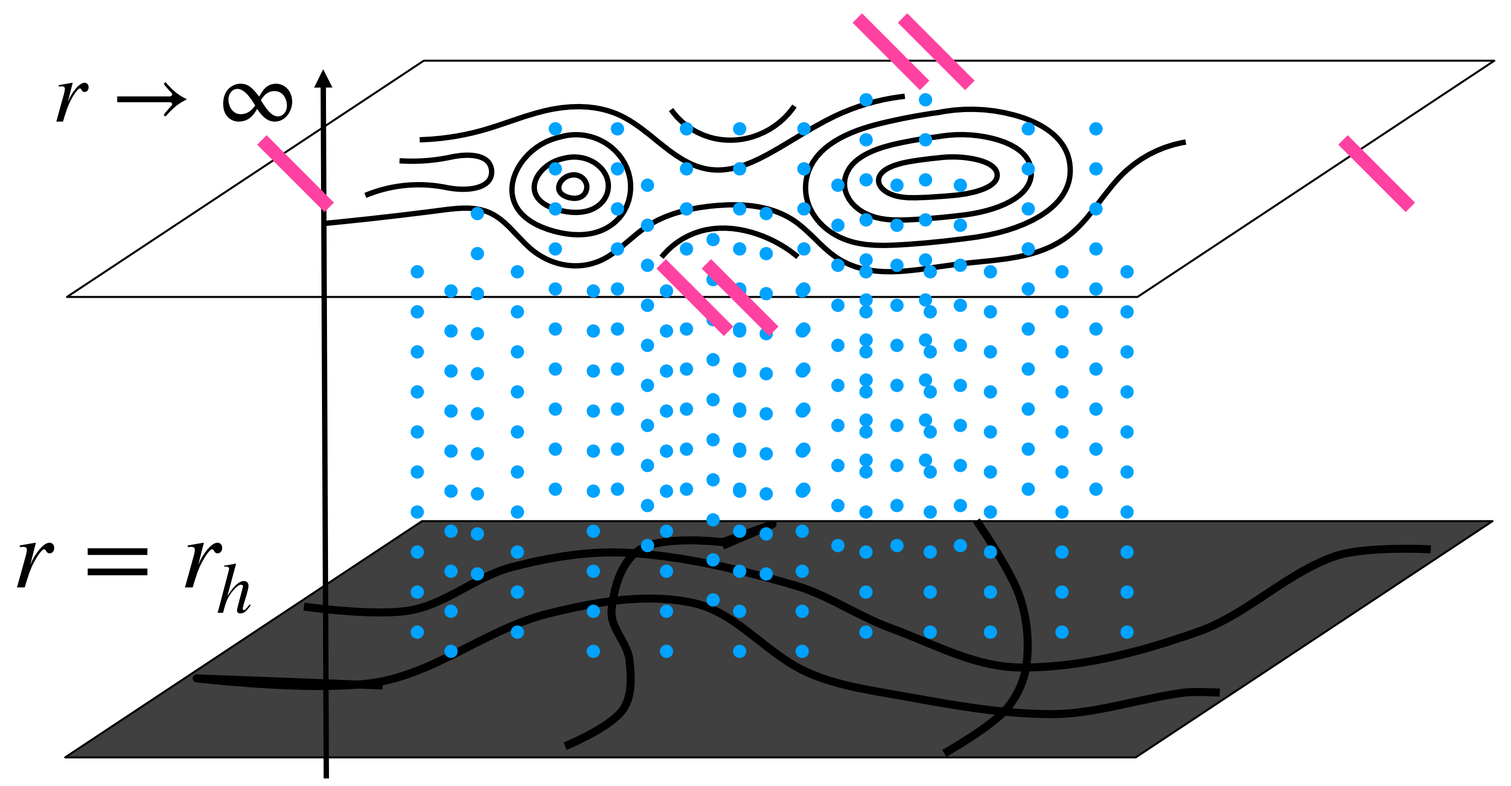




Stochastic gravity and turbulence

$\overline{T_{\mu\nu}}$





$$\overline{\xi(t, \vec{x})} = 0$$

$$\overline{\xi(t, \vec{x})\xi(t', \vec{x}')} = D(\vec{x} - \vec{x}')\delta(t - t')$$

$$\hat{D}(\vec{k}) = \delta(|\vec{k}| - k_f)$$

Holographic and turbulence

$$\overline{T_{\mu\nu}}$$

Recall:

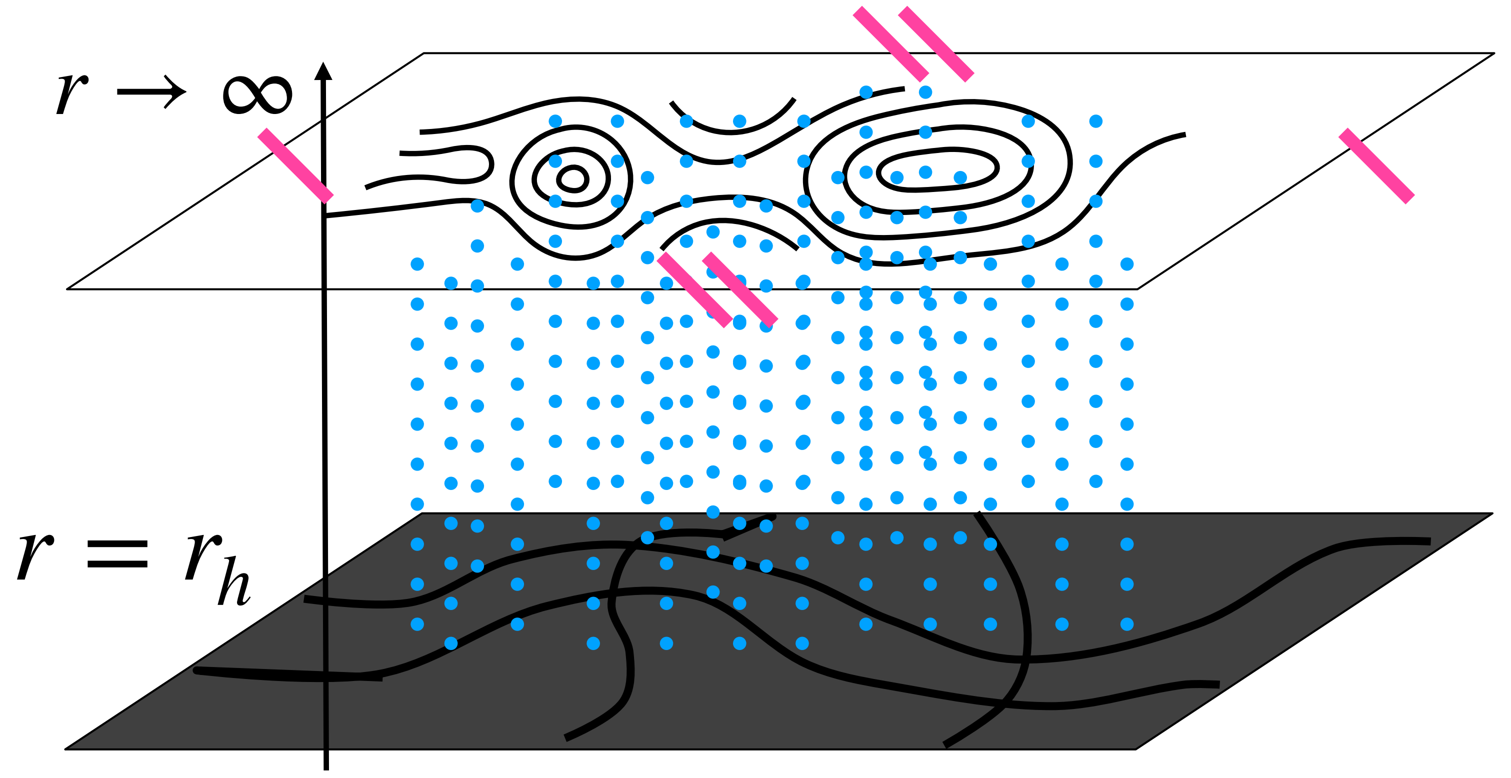
$$\hat{\epsilon} = \int \overline{\frac{1}{2}\rho \,|\,\hat{v}\,|^2 k d\theta_k} \propto k^{-\frac{5}{3}}$$

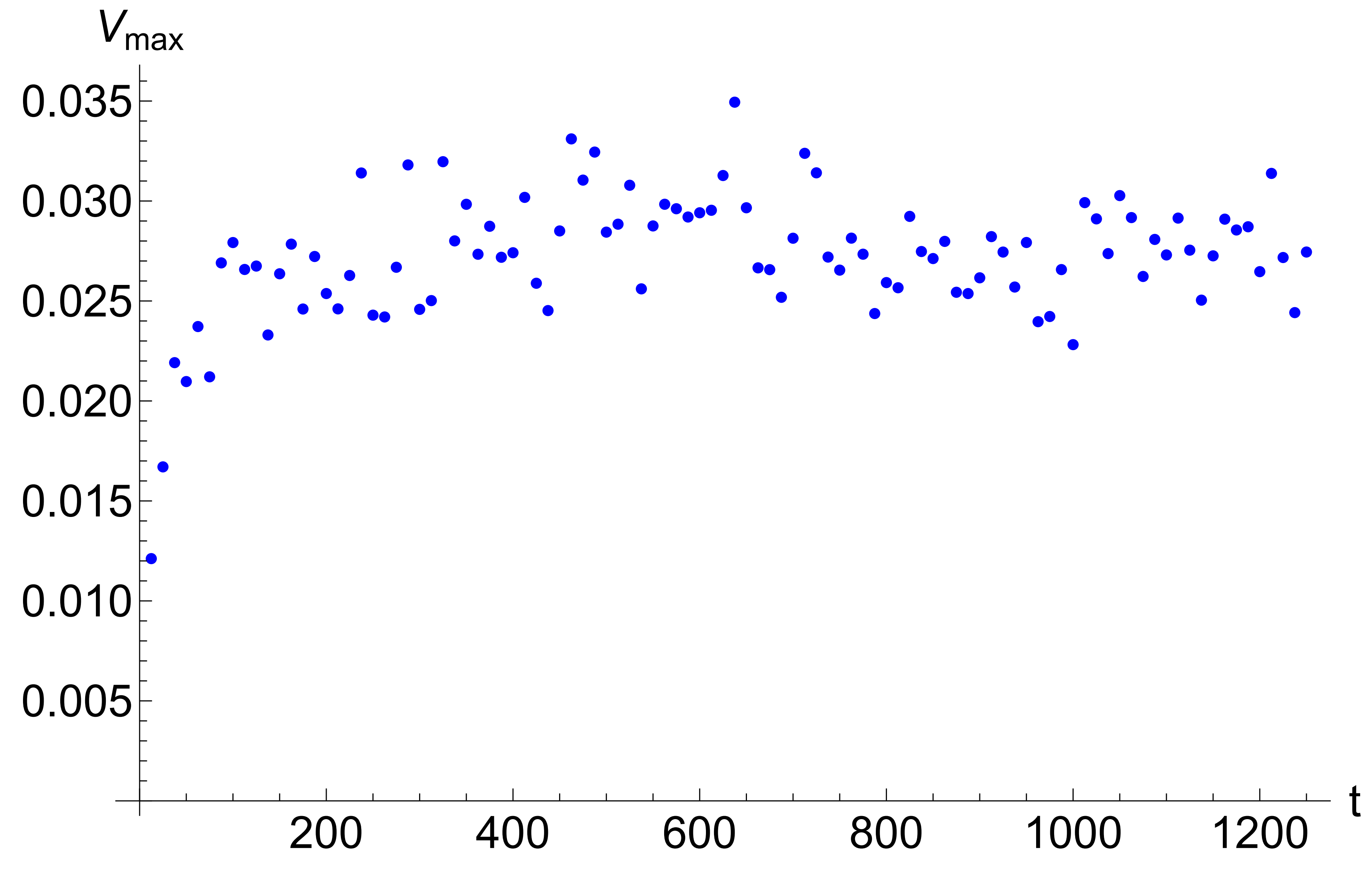
Define:

$$T^{\mu}{}_{\nu} u^{\nu} = -\,\epsilon u^{\nu}$$

with

$$u^{\mu} = \gamma\left(1, \overrightarrow{v}\right)$$

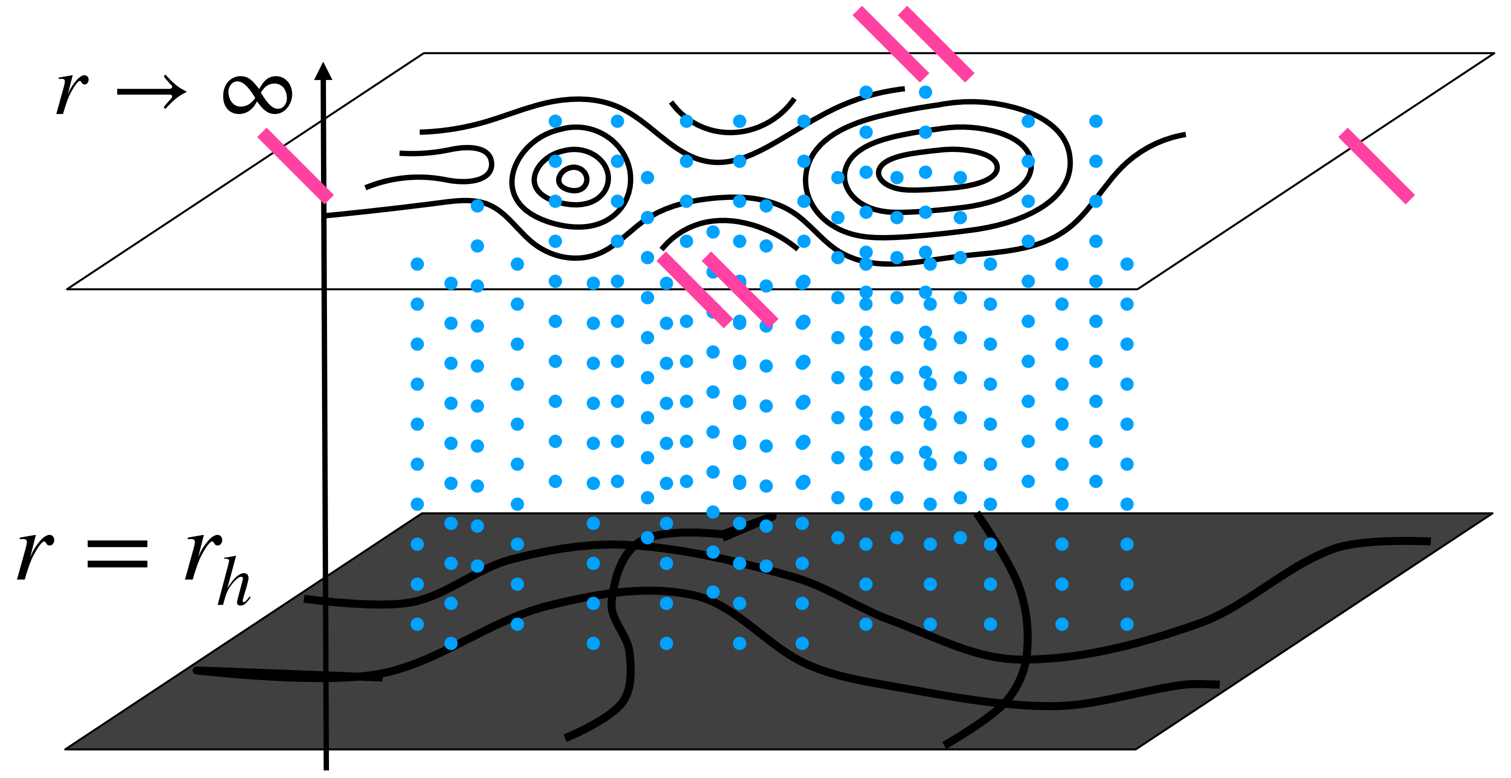




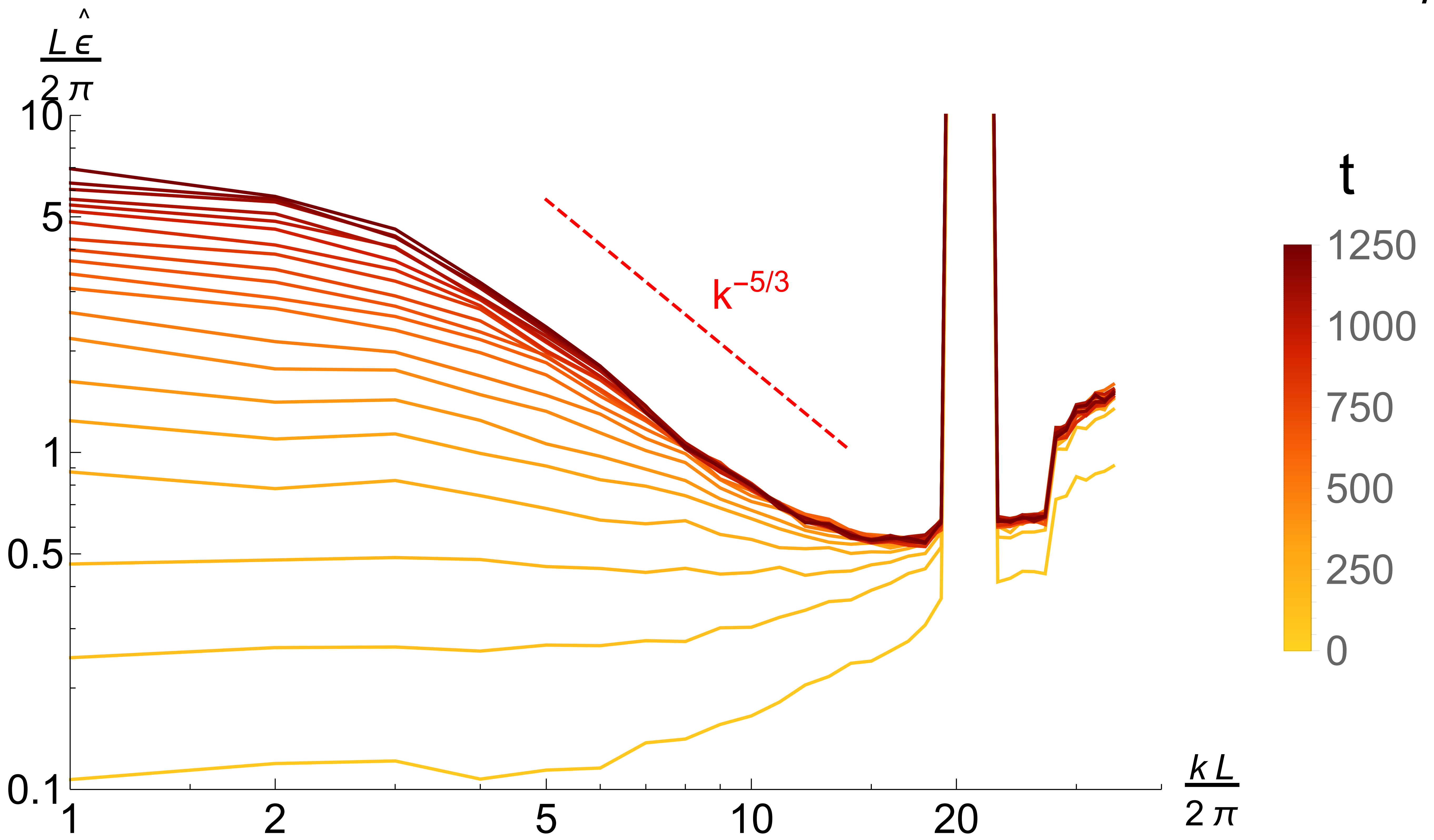
Holographic and turbulence

Recall:

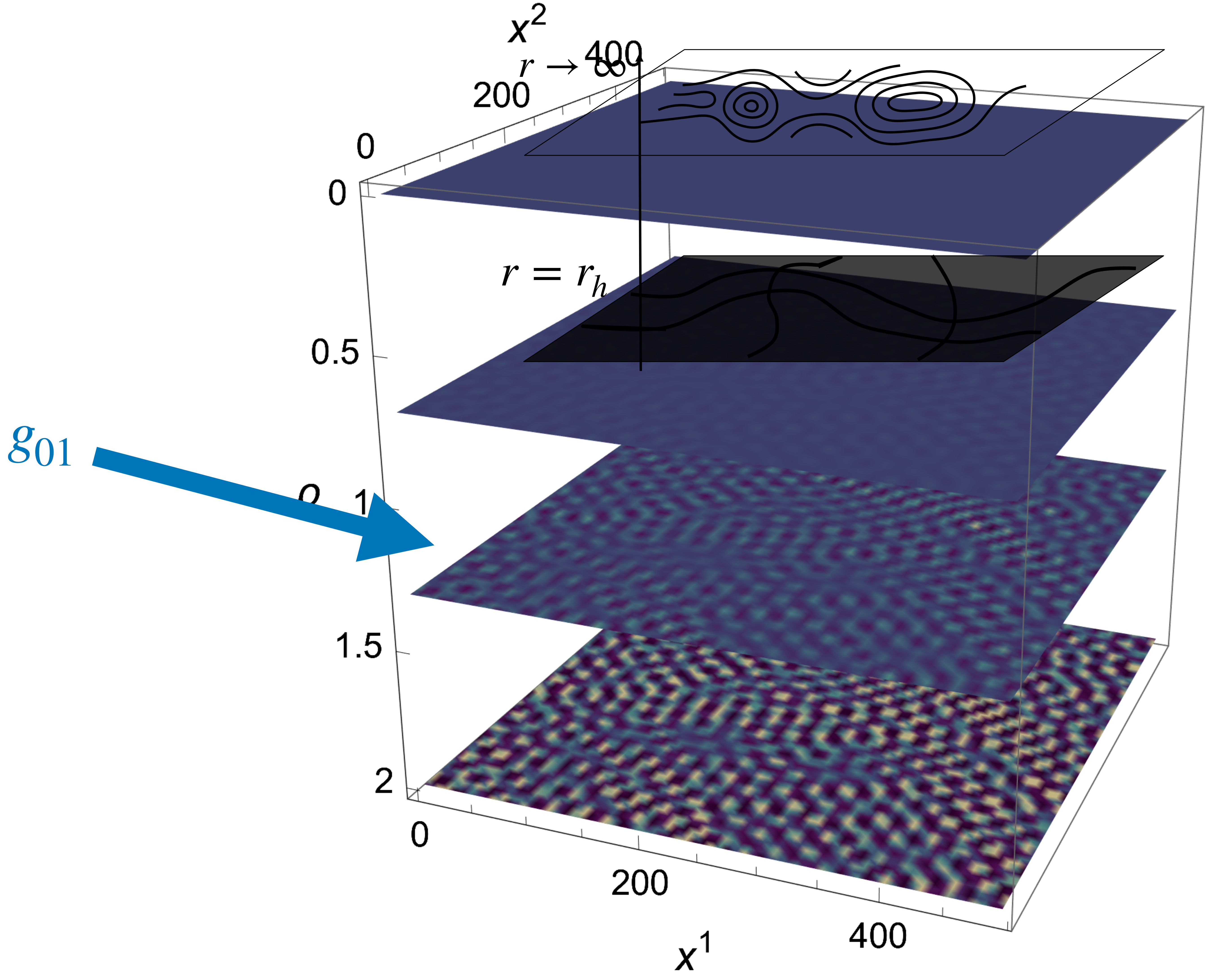
$$\hat{\epsilon} = \int \overline{\frac{1}{2}\rho |\hat{v}|^2 k d\theta_k} \propto k^{-\frac{5}{3}}$$



We find:



Holographic and turbulence



Holographic and turbulence

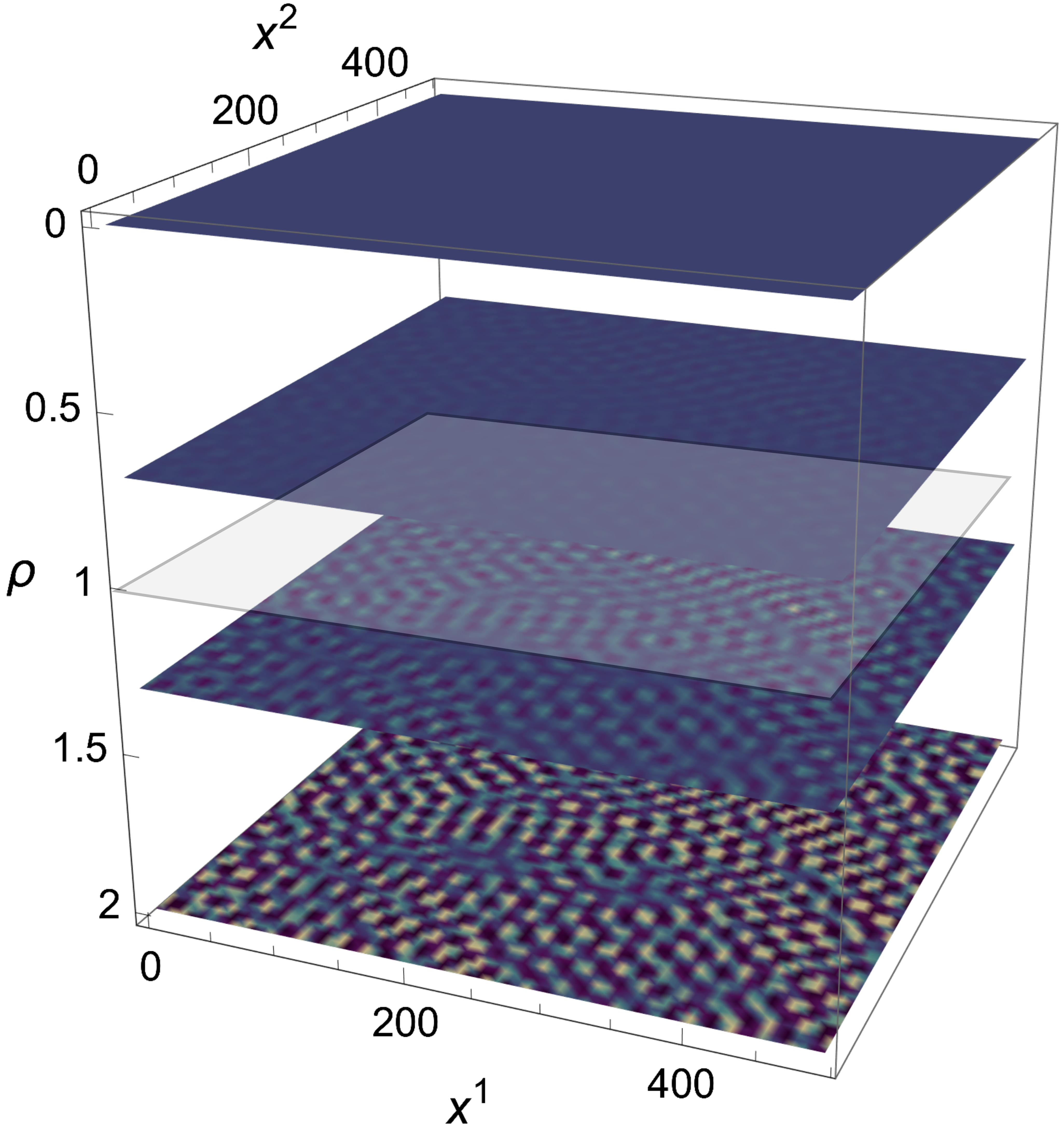
There's an apparent horizon at

$$0.9 \leq \rho = \rho_h(t, x_1, x_2) \leq 1.1$$

We would like to find a geometric quantity that encodes

$$\overline{\left((\vec{v}(\vec{r}) - \vec{v}(0)) \cdot \hat{r}\right)^n} \propto |r|^{\zeta_n}$$

(Recall that $\zeta_n = n/3$ for Kolmogorov theory)



Holographic and turbulence

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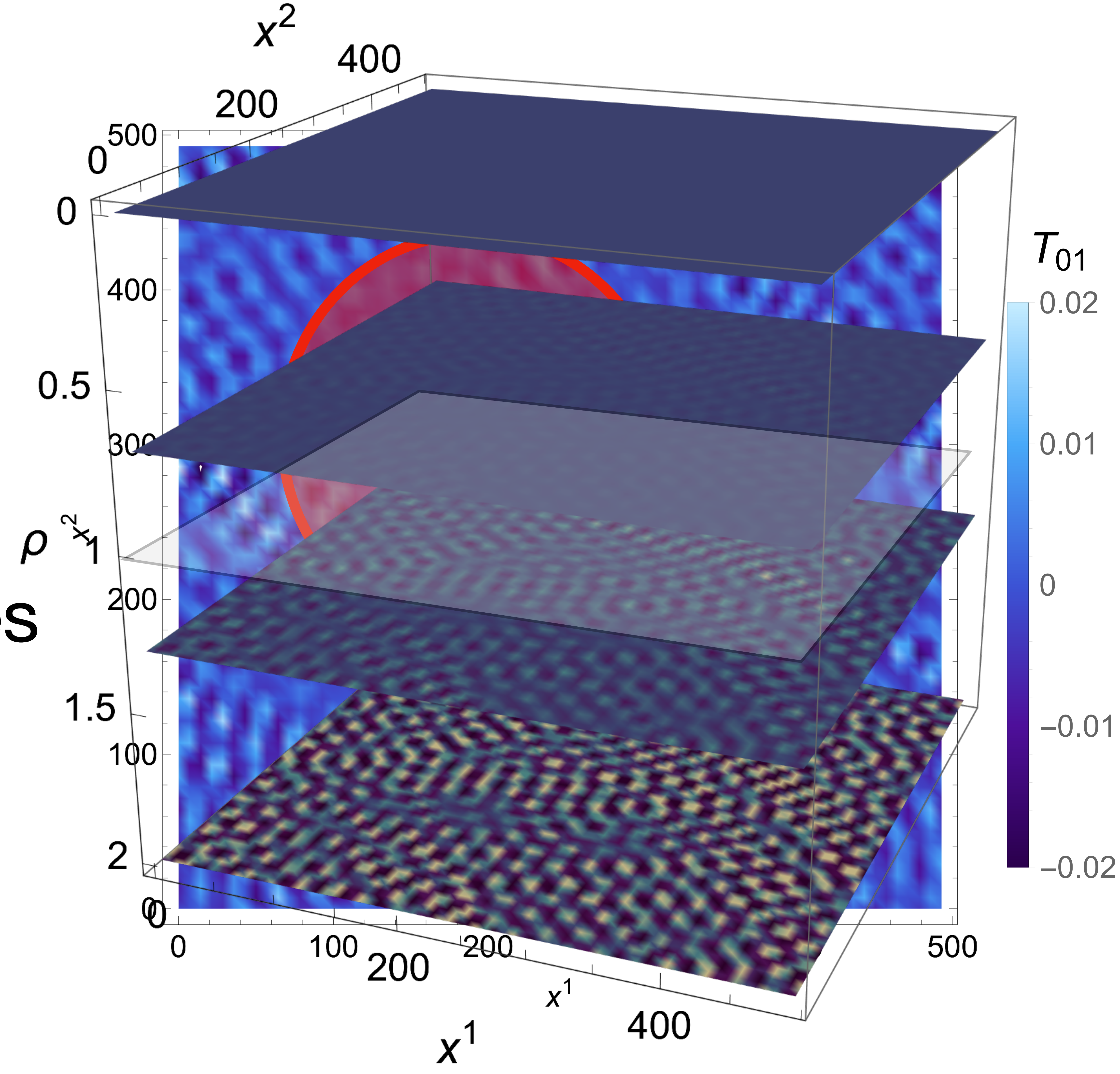
An alternate expression which encodes ζ_n is

$$\epsilon_R(x) = \frac{1}{V_R} \int_{|x-x'|\leq R} \left(\partial_i v_j + \partial_j v_i\right)^2 d^d x'$$



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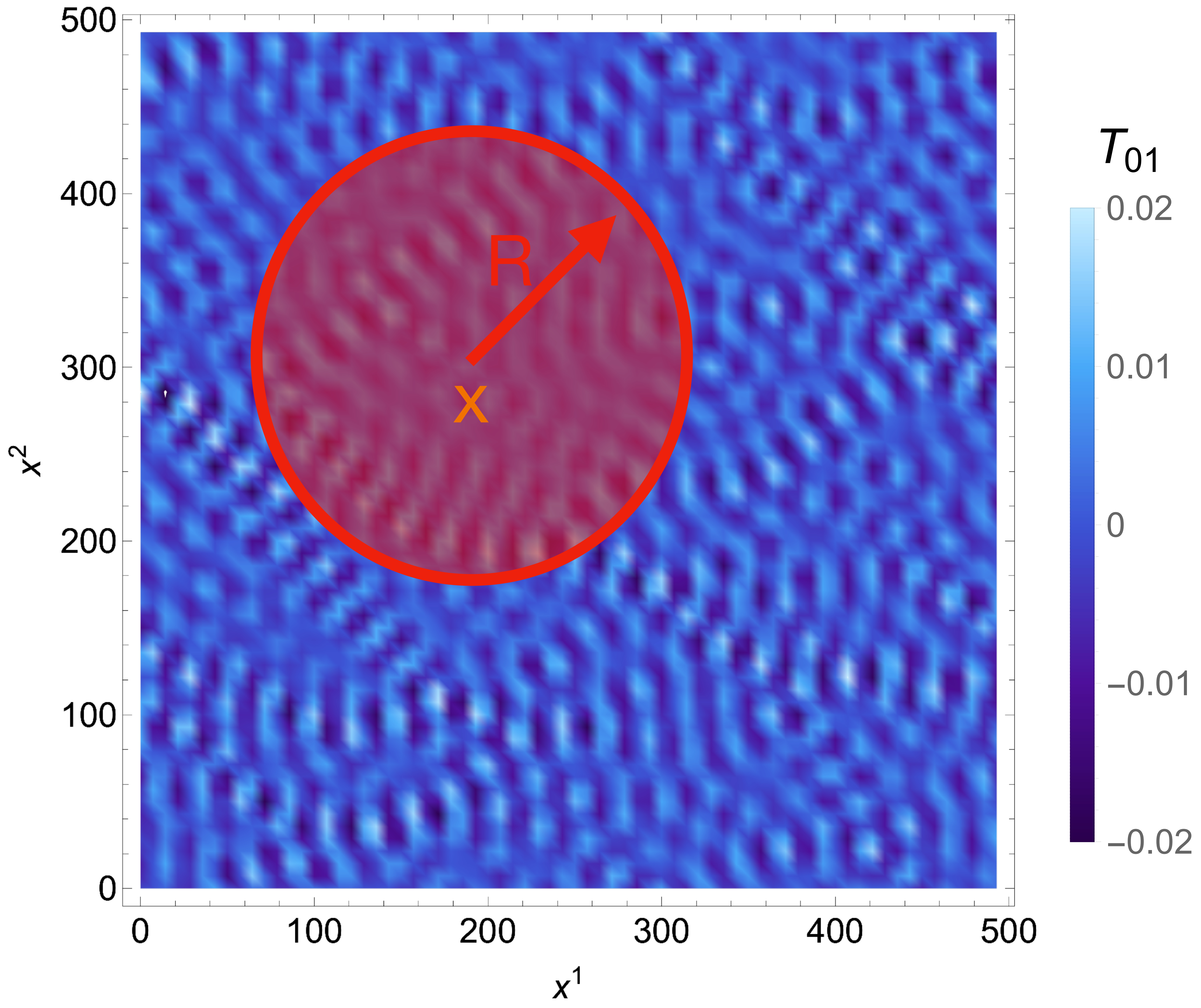

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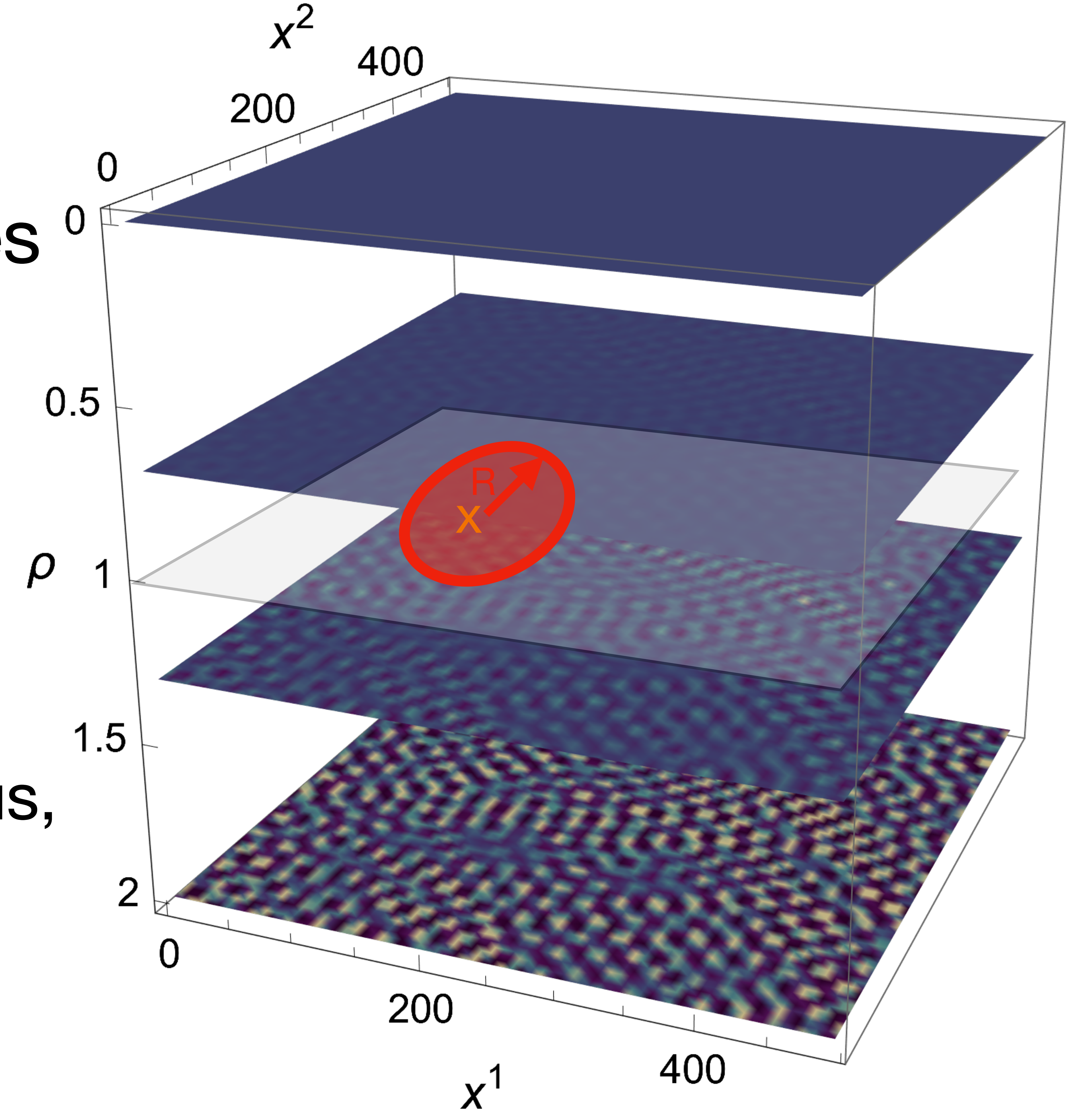
As it turns out, the extrinsic curvature of the horizon, K_{ij}, is proportional to the shear. Thus,

$$\epsilon_R(x) \sim e_R(x)$$

$$e_R(x) = \frac{1}{D_R}\int_{|x-x'|\leq R} K_i^{\ j} K_j^{\ i} d^d x'$$



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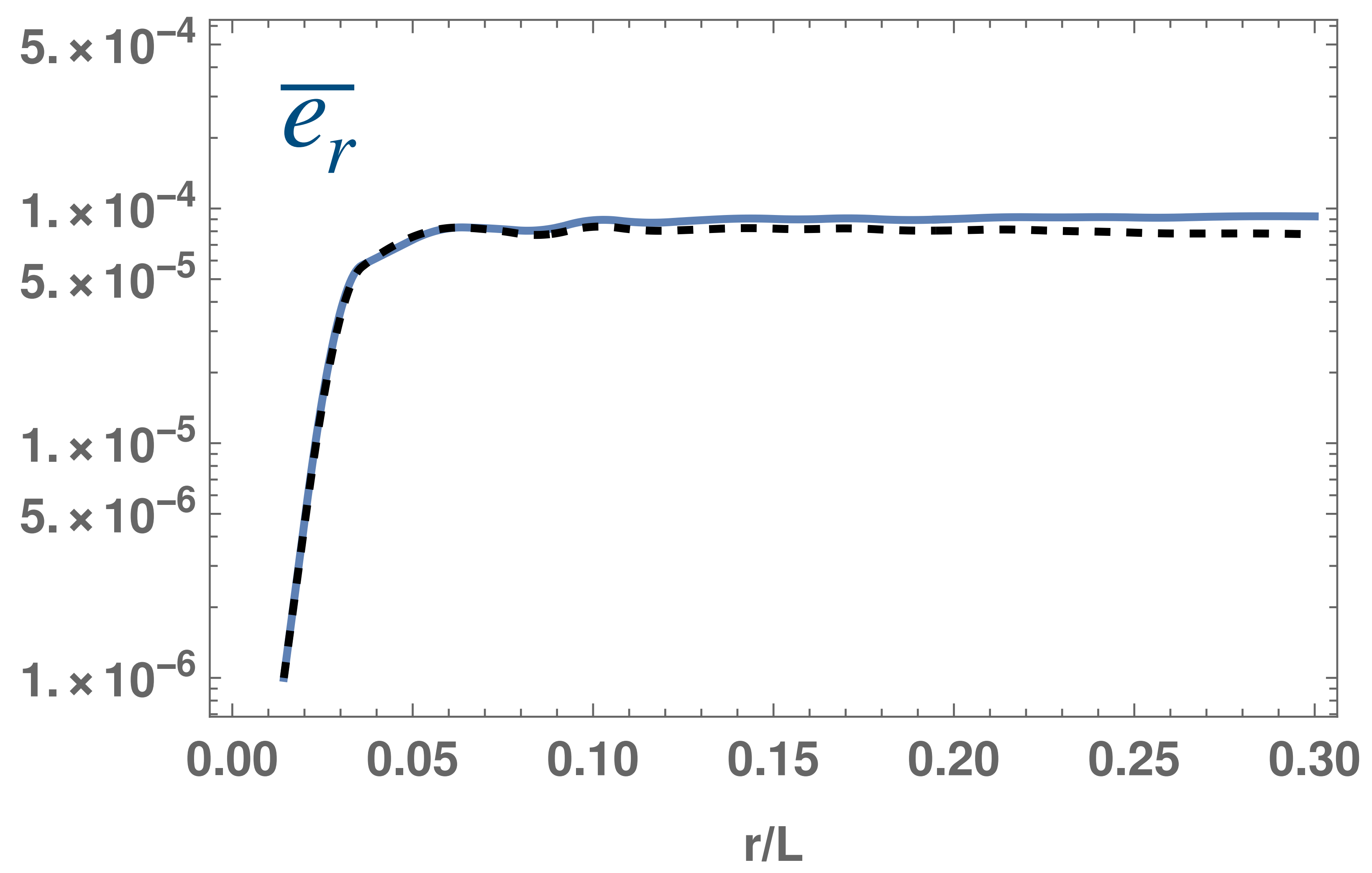
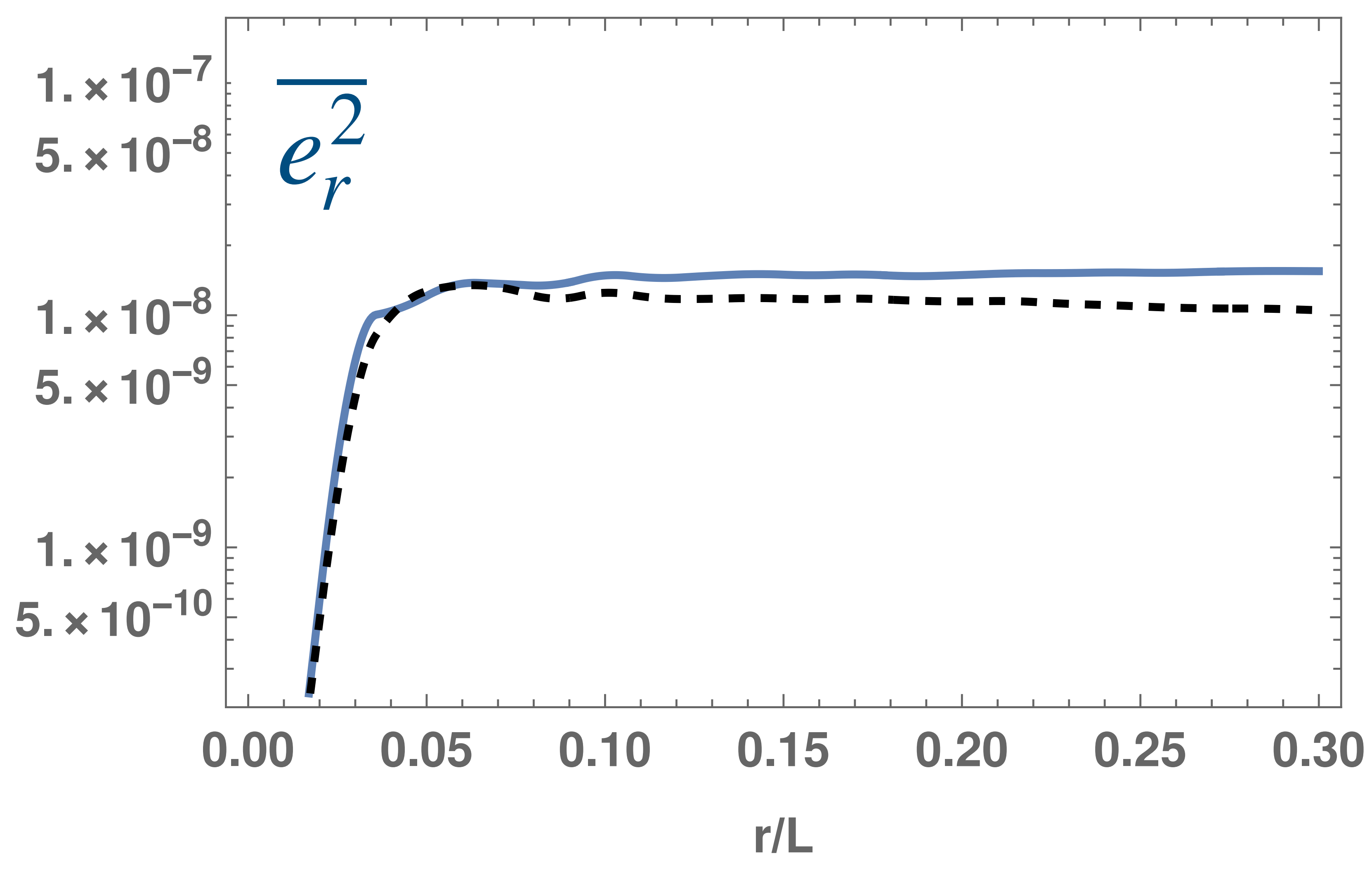
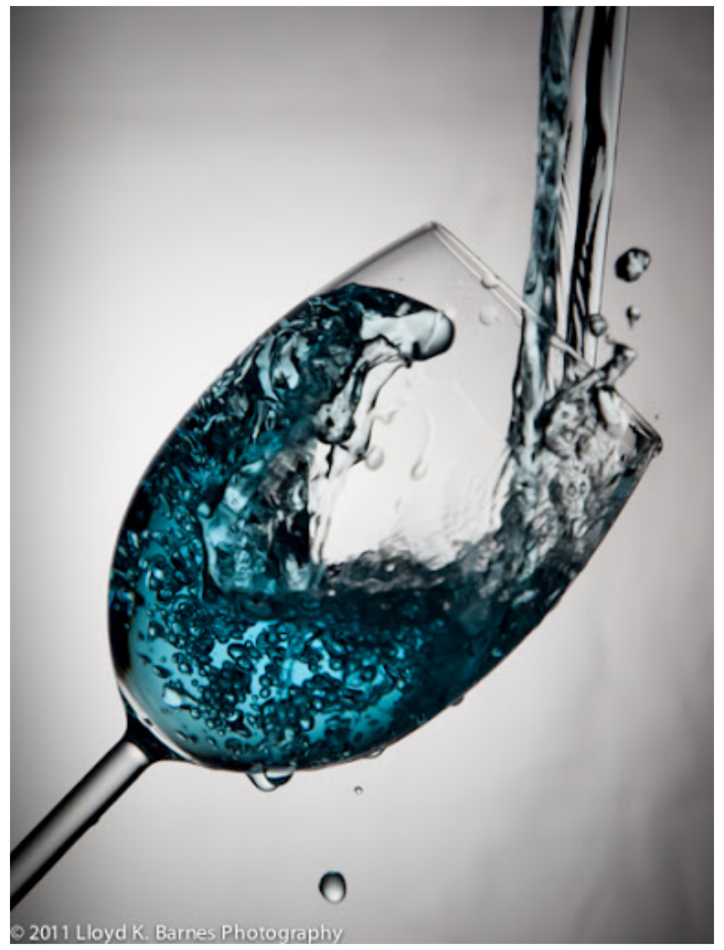
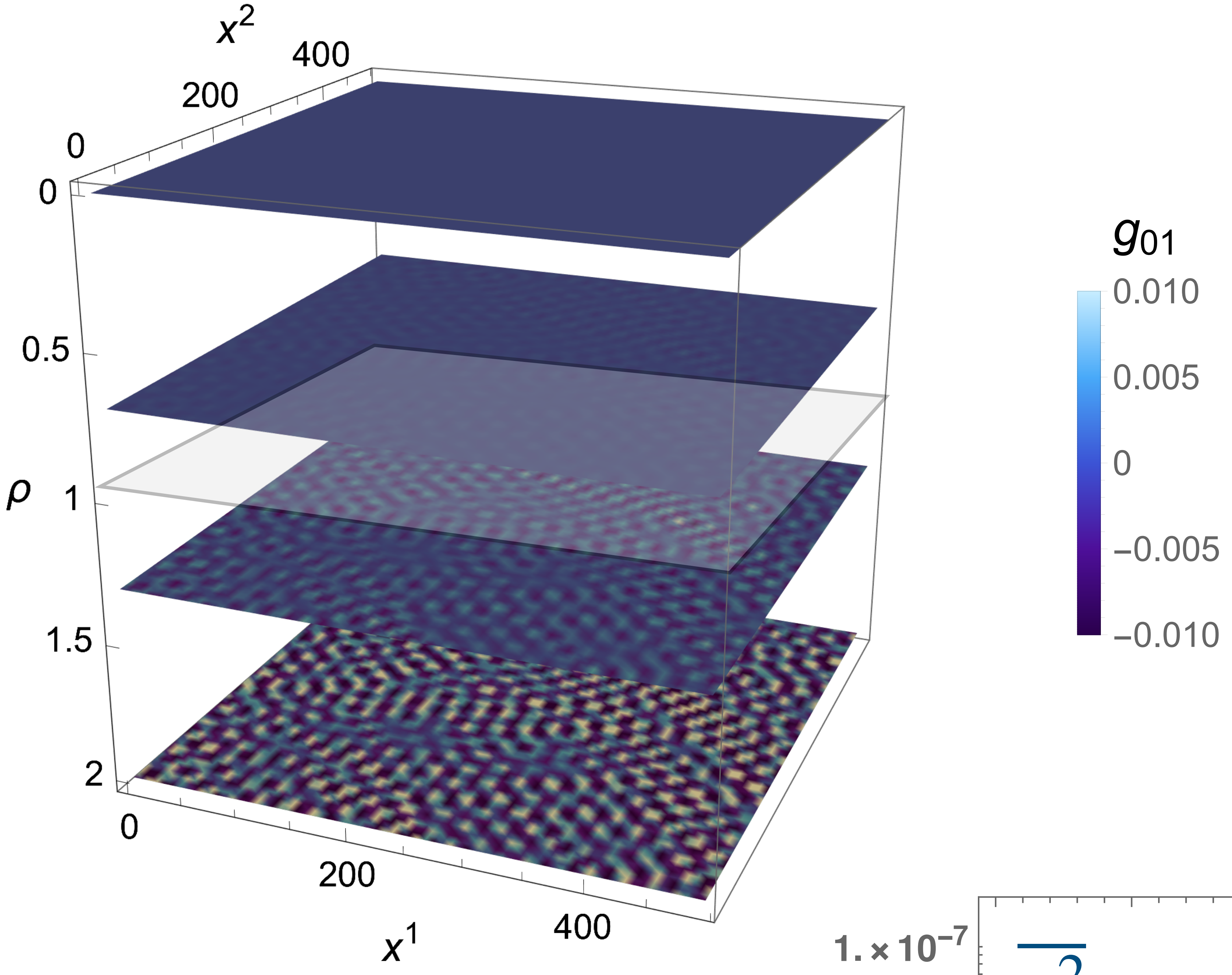
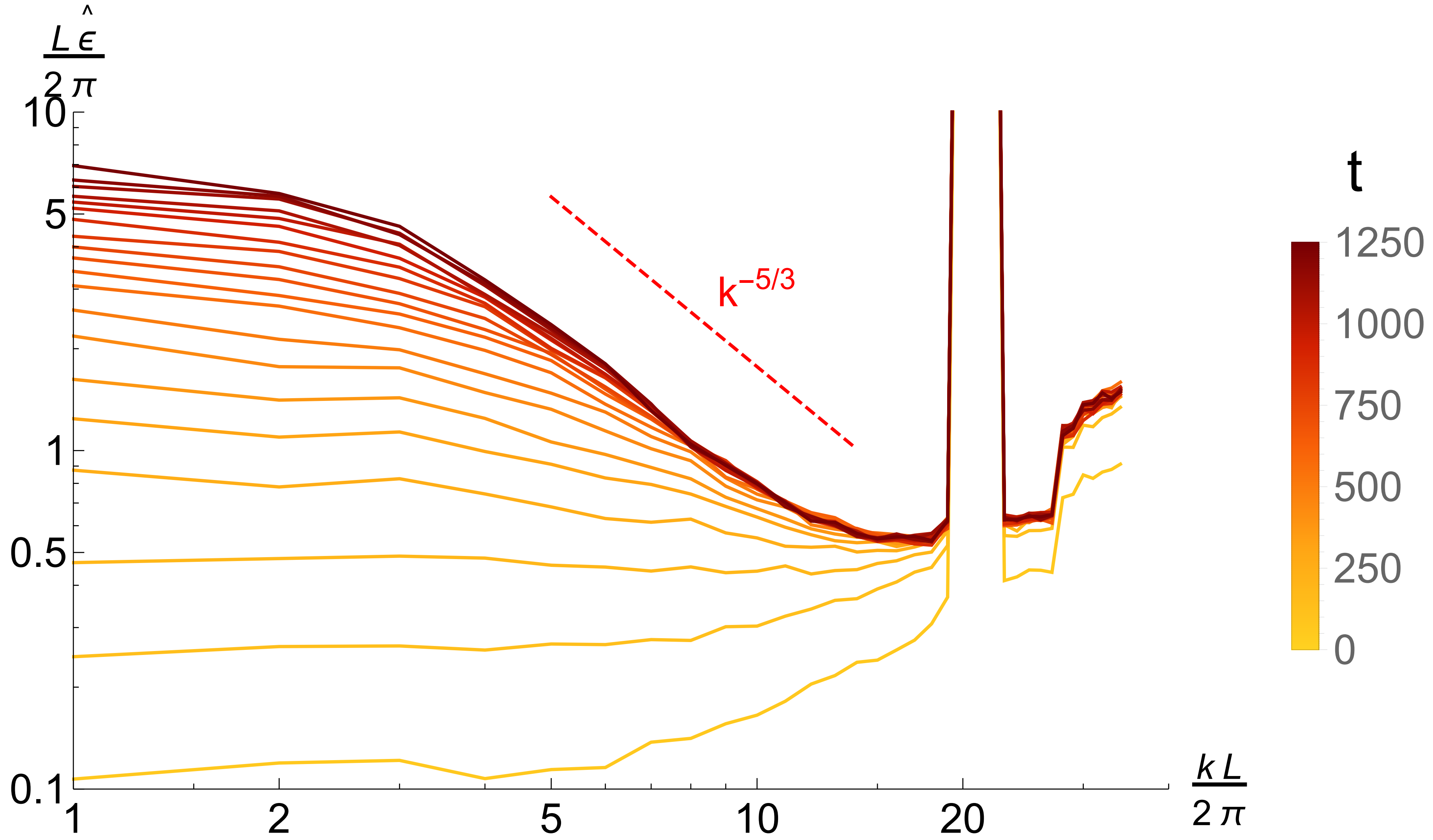


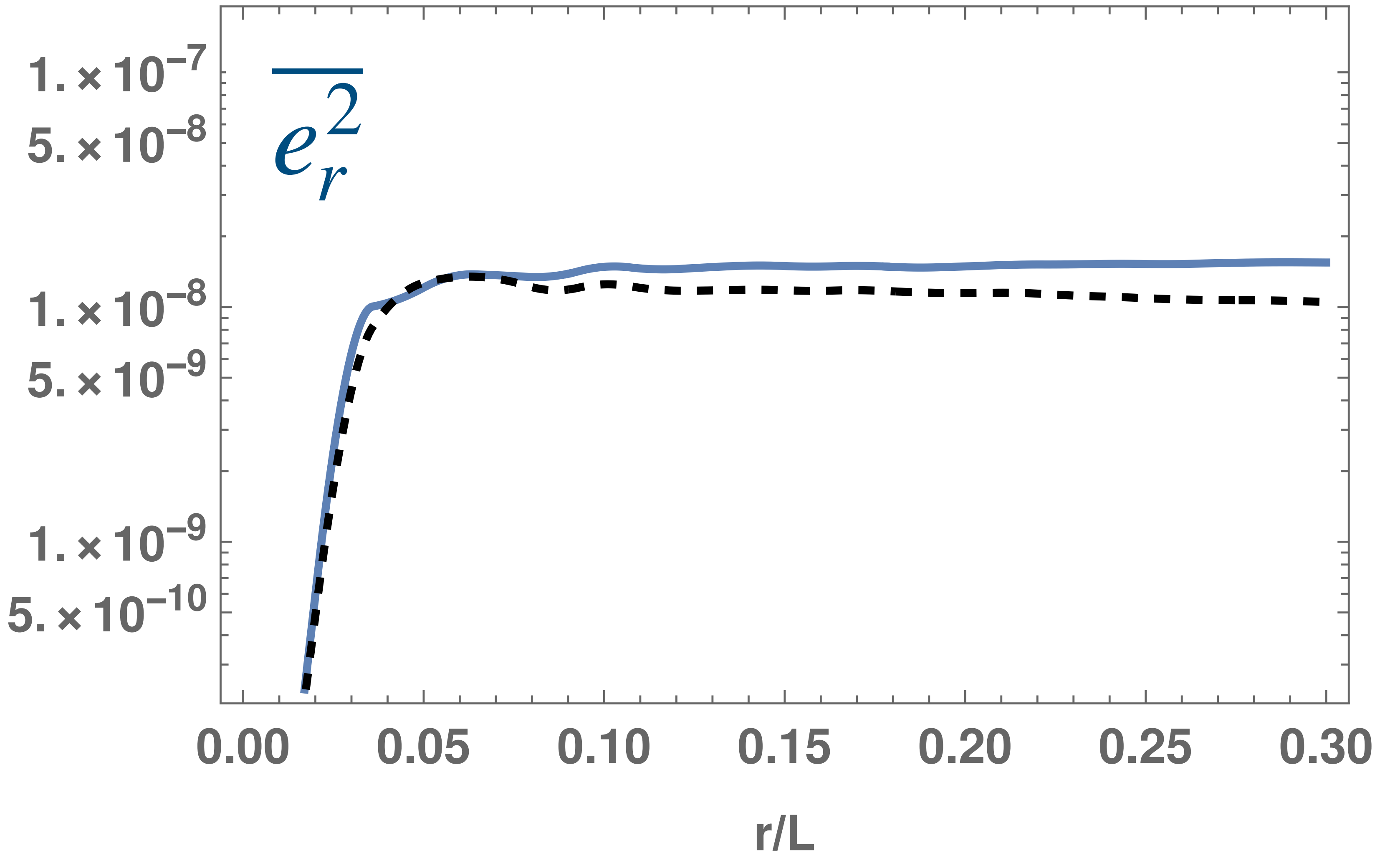


Summary

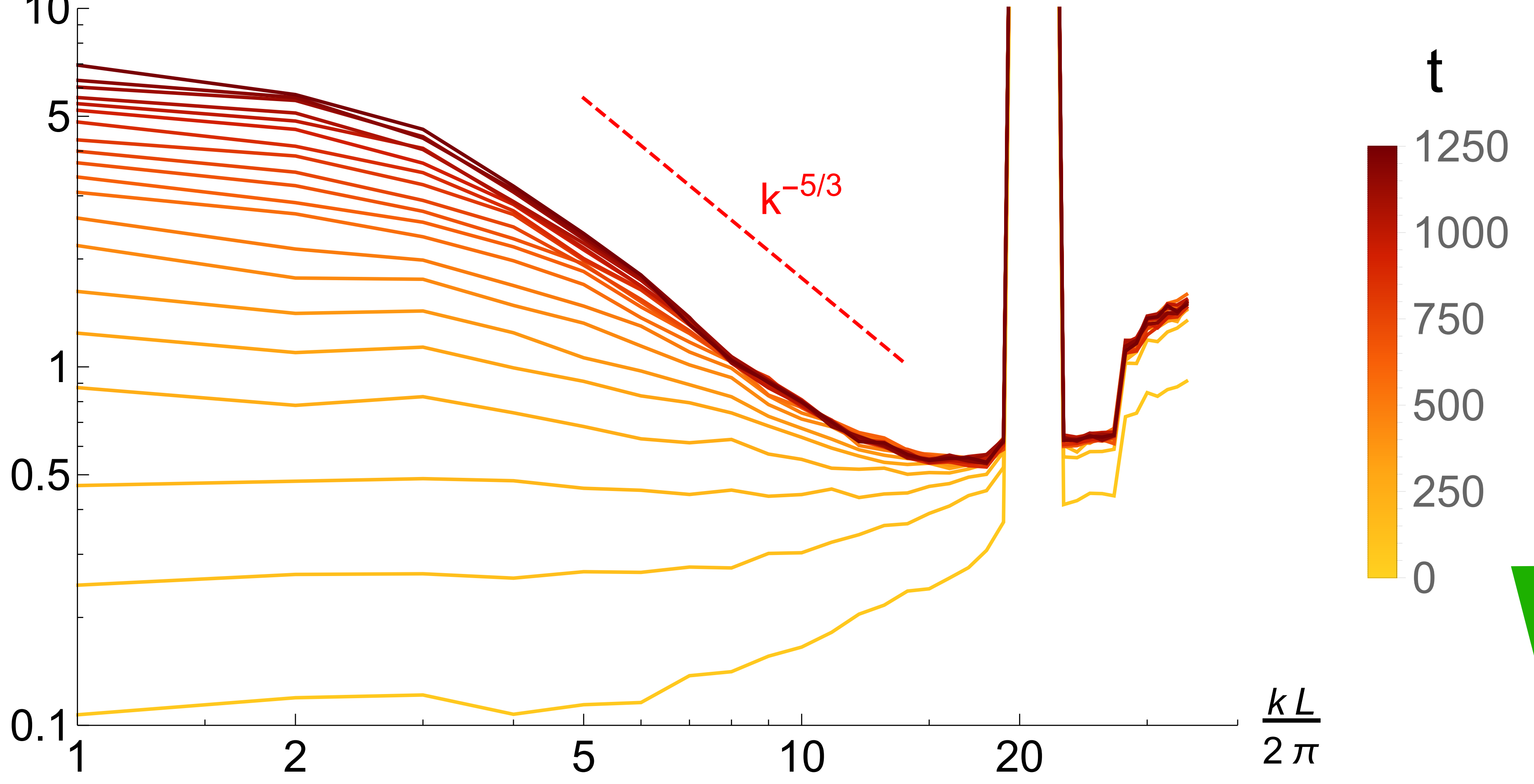

© 2011 Lloyd K. Barnes Photography

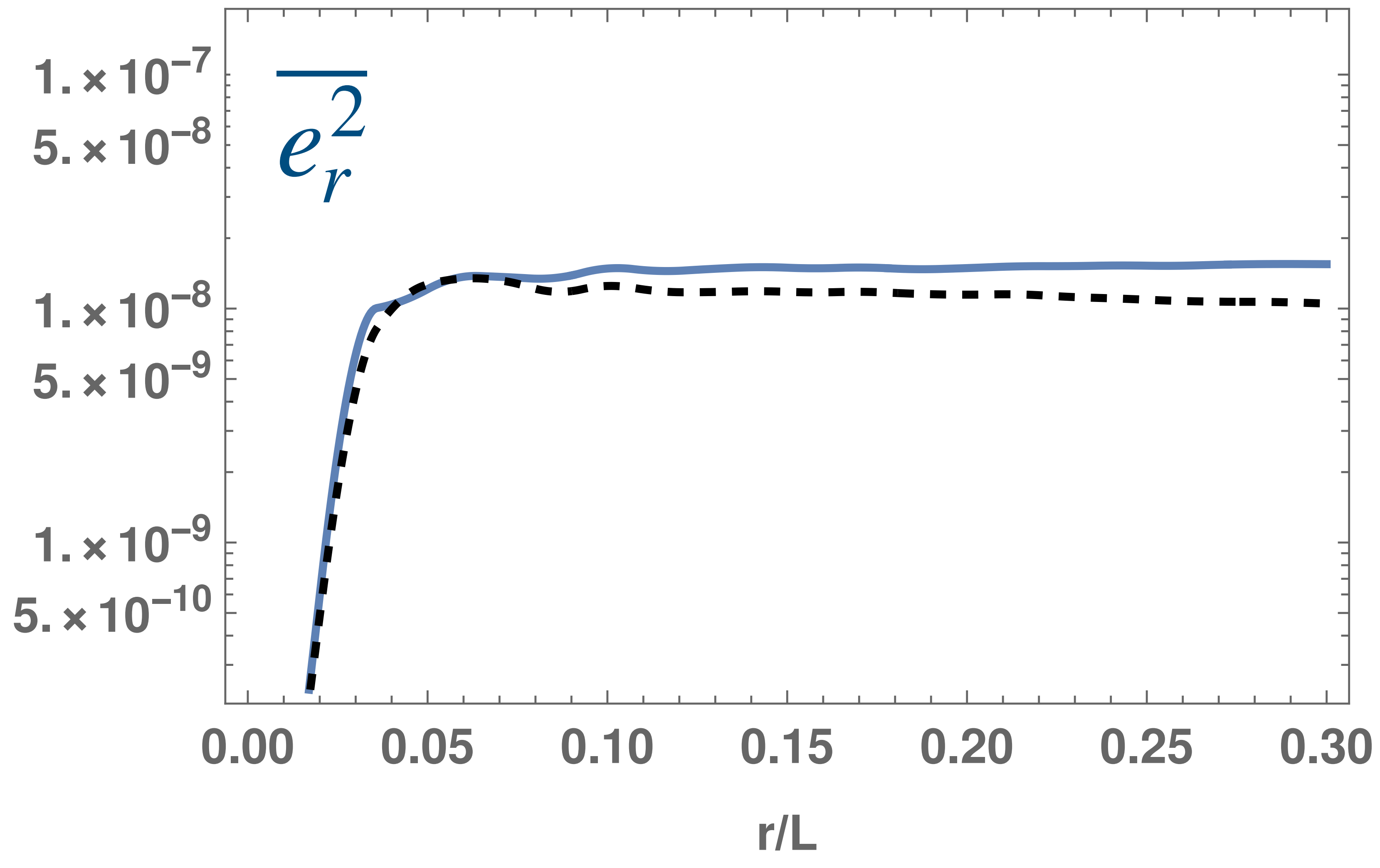

x^2
0
200
400
ρ
0
0.5
1
1.5
2
0
200
400
x^1
g_{01}
0.010
0.005
0
−0.005
−0.010


$\frac{L\hat{\epsilon}}{2\pi}$
10
5
1
0.5
0.1
$k^{-5/3}$
1
2
5
10
20
$\frac{kL}{2\pi}$
t
1250
1000
750
500
250
0

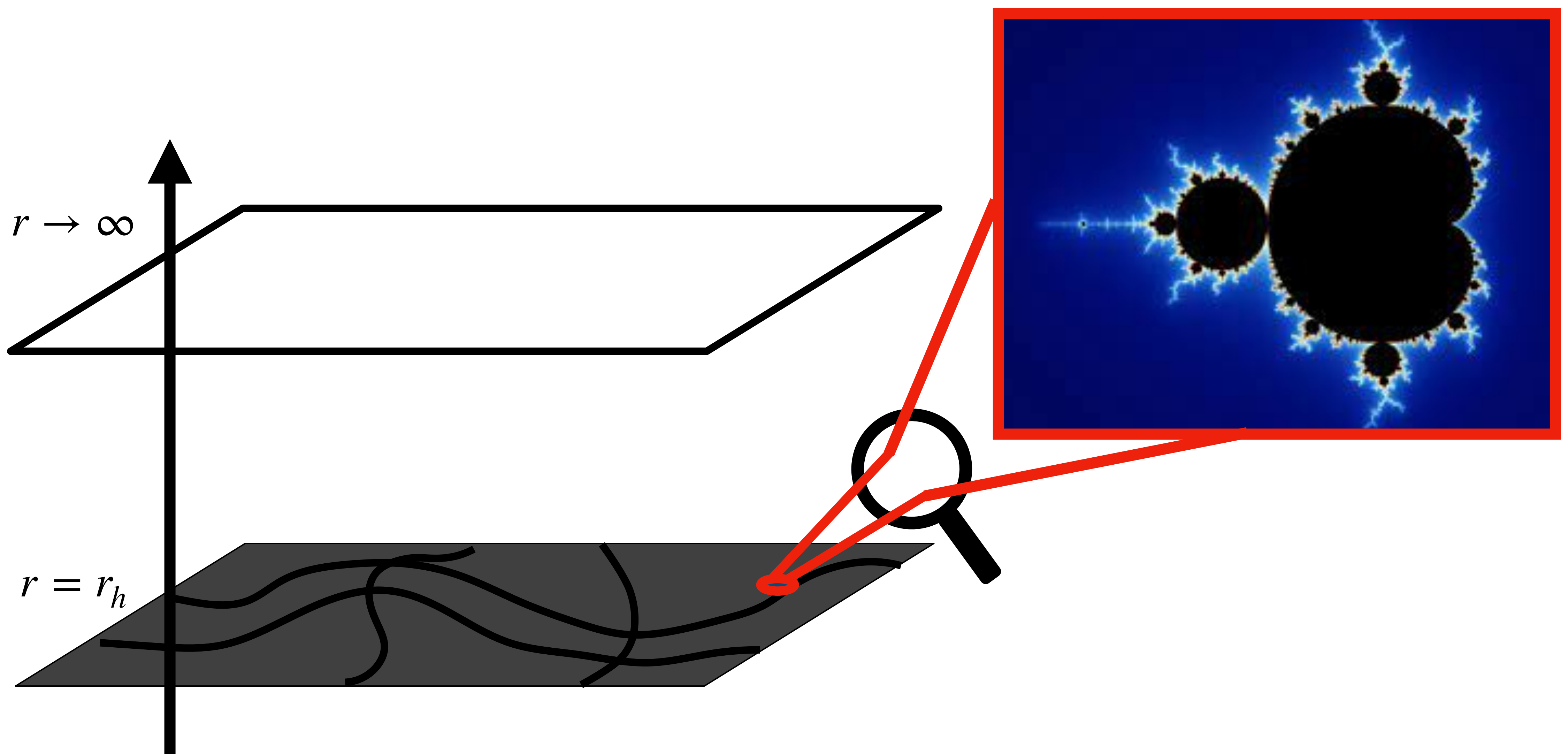

$\overline{e_r^2}$
$1.\times10^{-7}$
$5.\times10^{-8}$
$1.\times10^{-8}$
$5.\times10^{-9}$
$1.\times10^{-9}$
$5.\times10^{-10}$
0.00
0.05
0.10
0.15
0.20
0.25
0.30
r/L

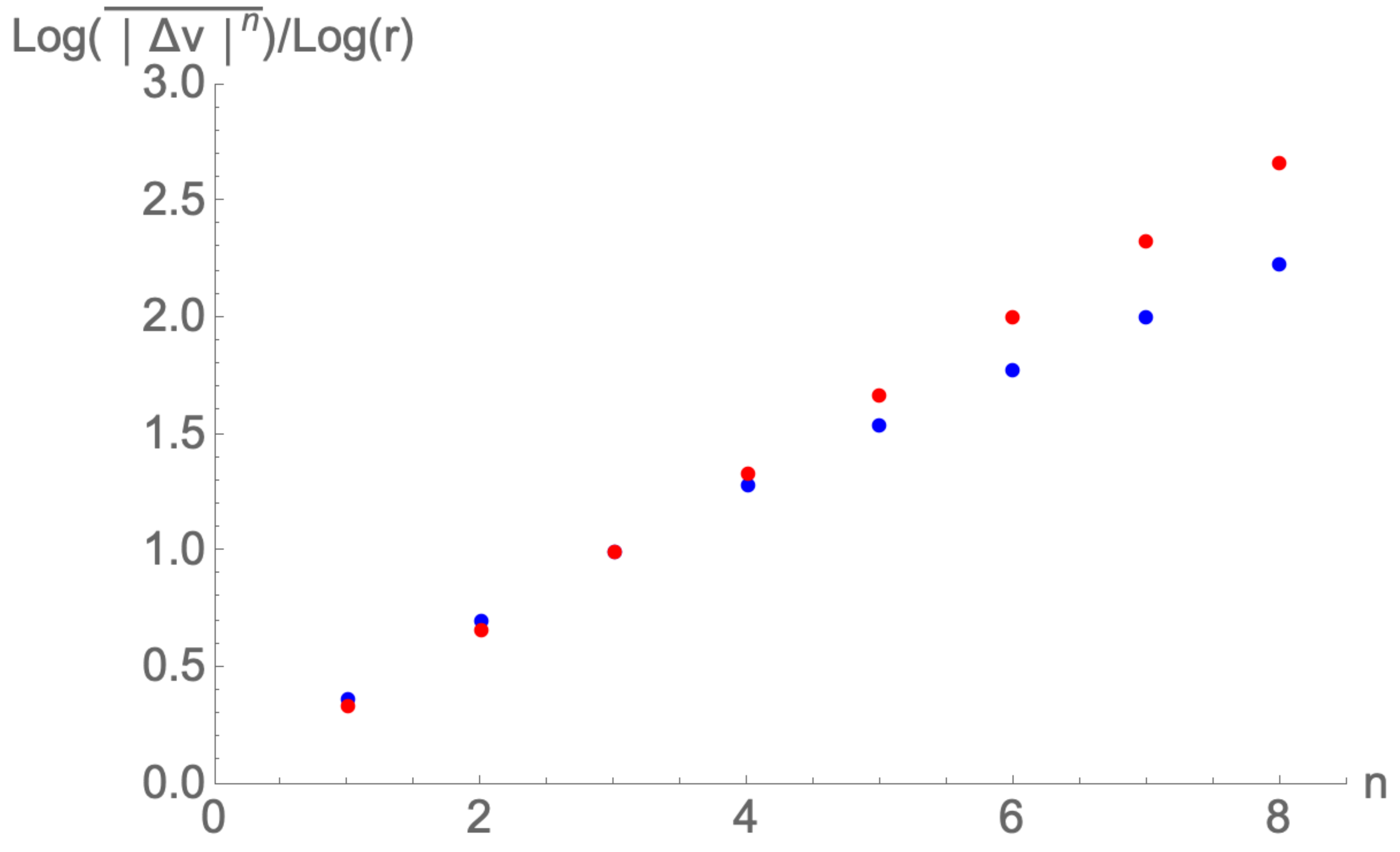
Outlook





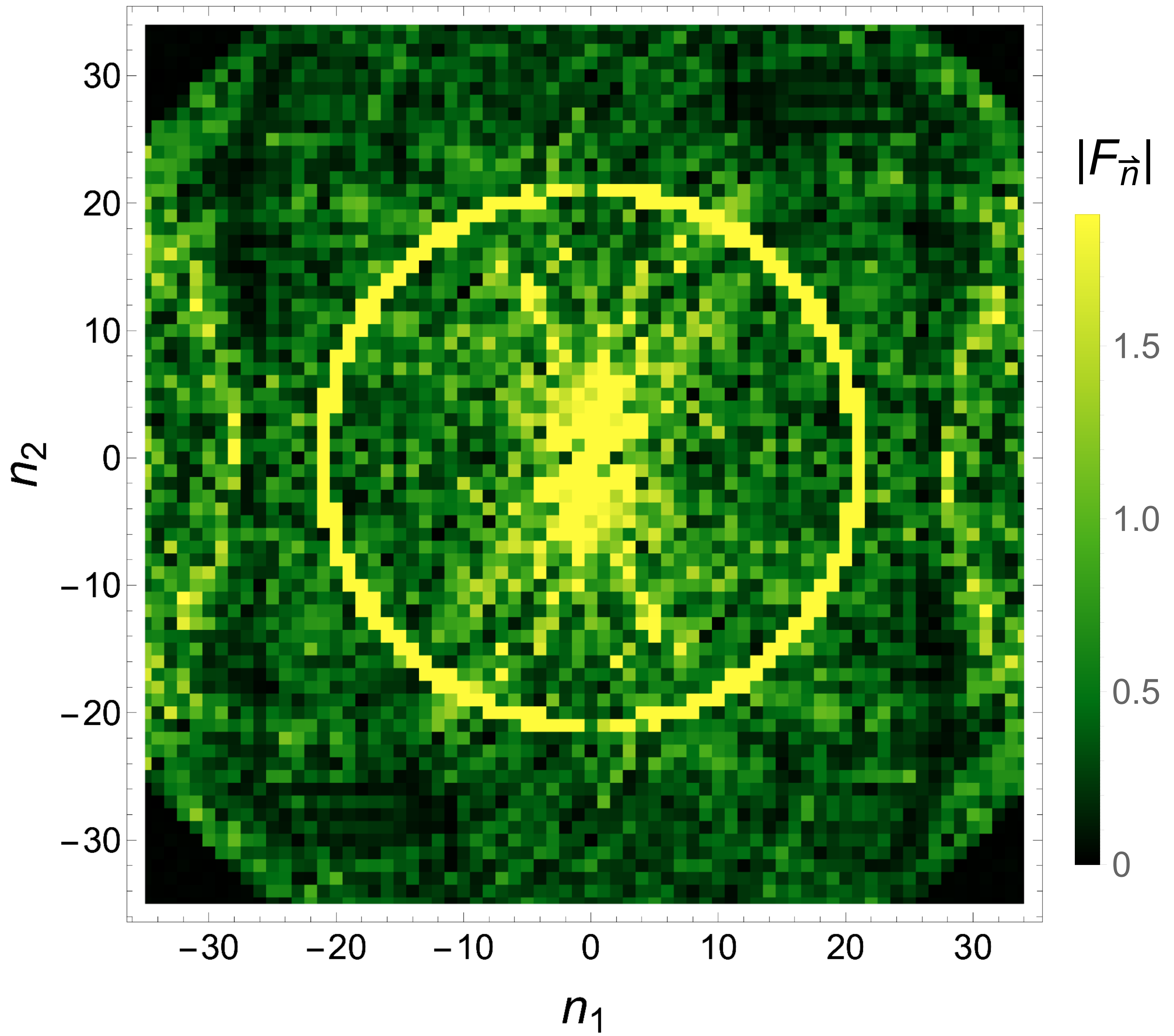
$$R_{mn} - \frac{1}{2} R g_{mn} - \frac{12}{\ell^2} g_{mn} = 0$$

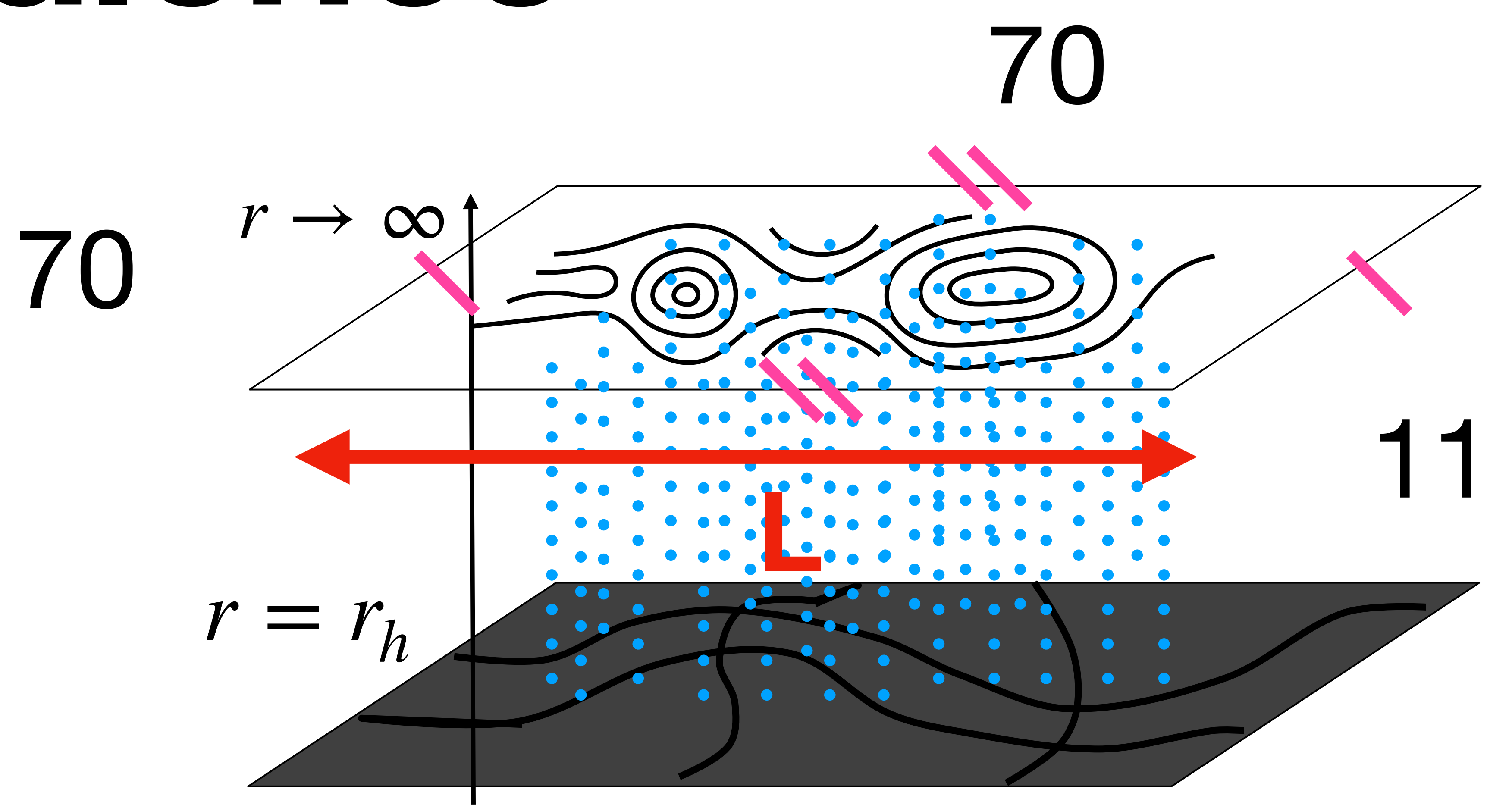




Stochastic gravity and turbulence

$$\overline{T_{\mu\nu}} = \overline{g^{(3)}_{\mu\nu}}$$





$$T_{01} = \frac{1}{L} \sum_{\vec{n}} F_{\vec{n}} e^{i \frac{2\pi \vec{n}}{L L} \vec{x}}$$

Stochastic gravity and turbulence

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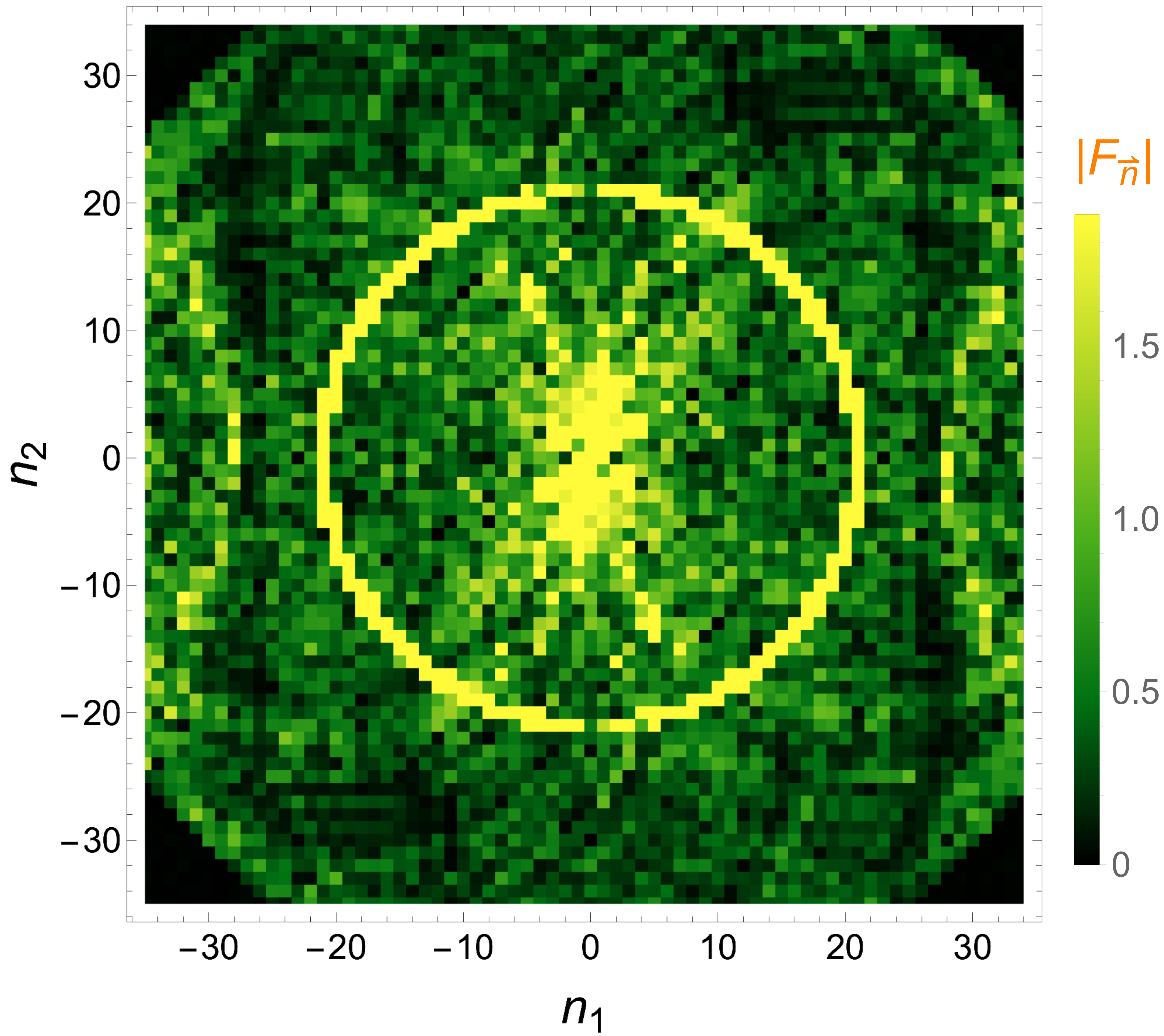

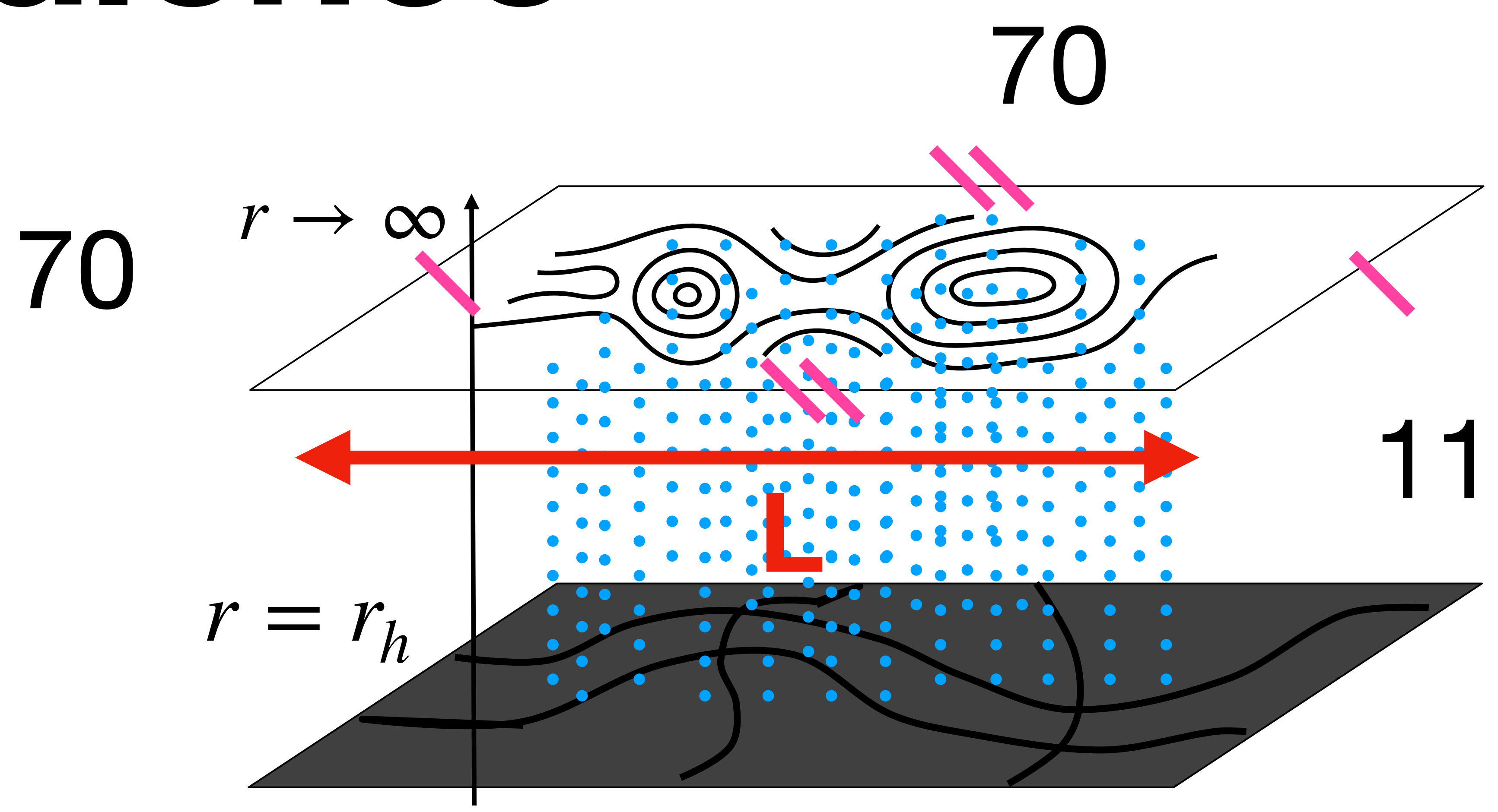

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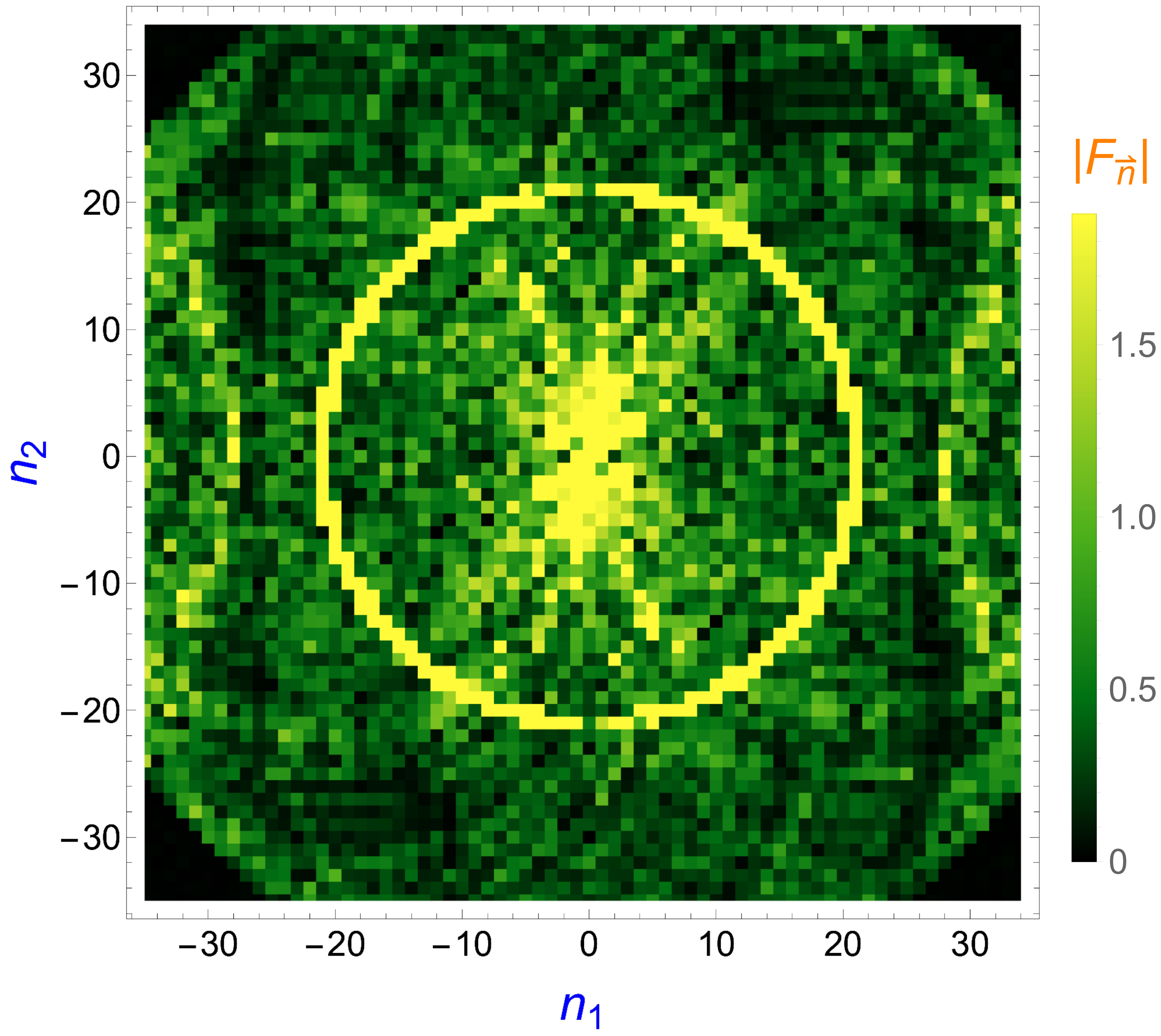

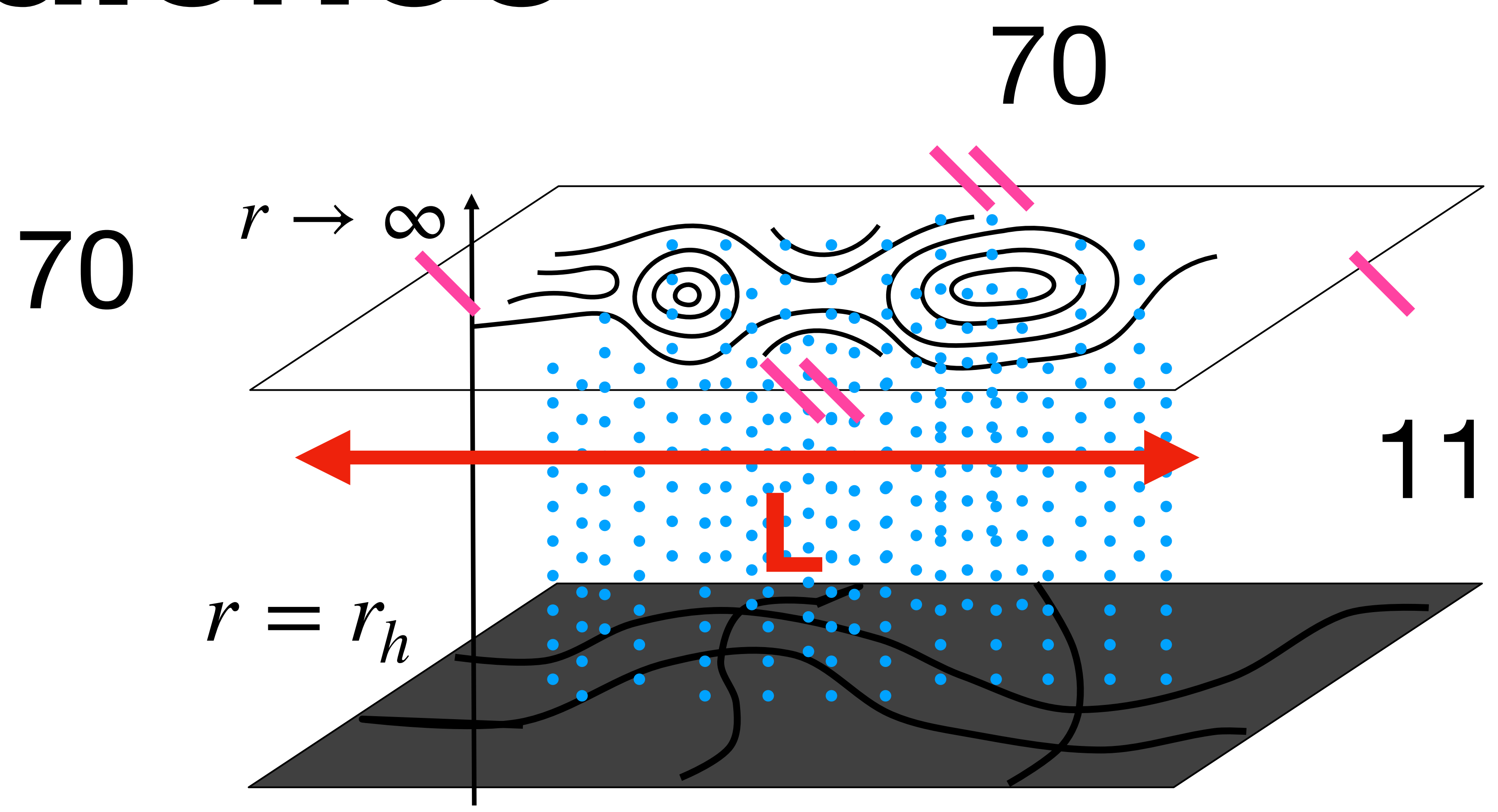

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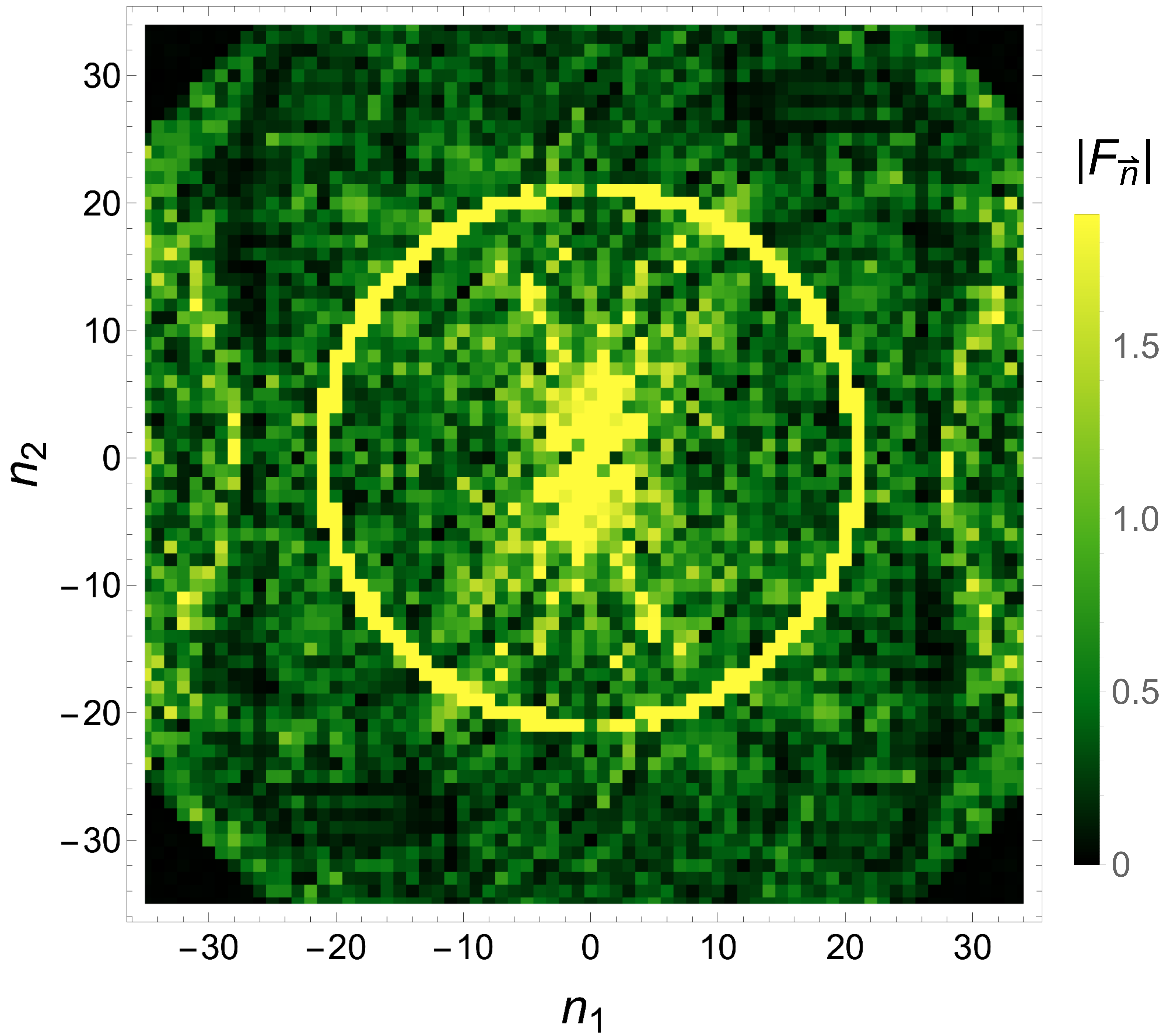

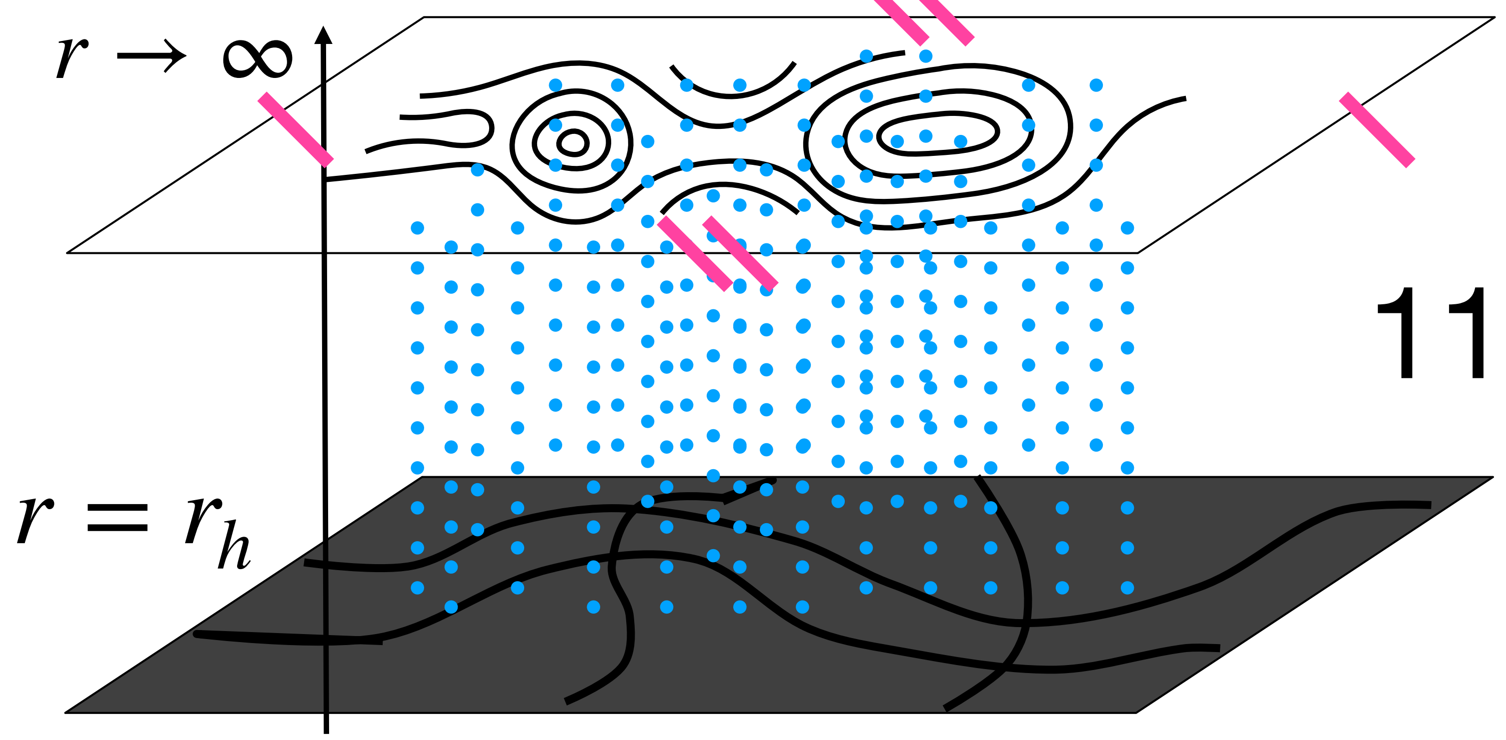

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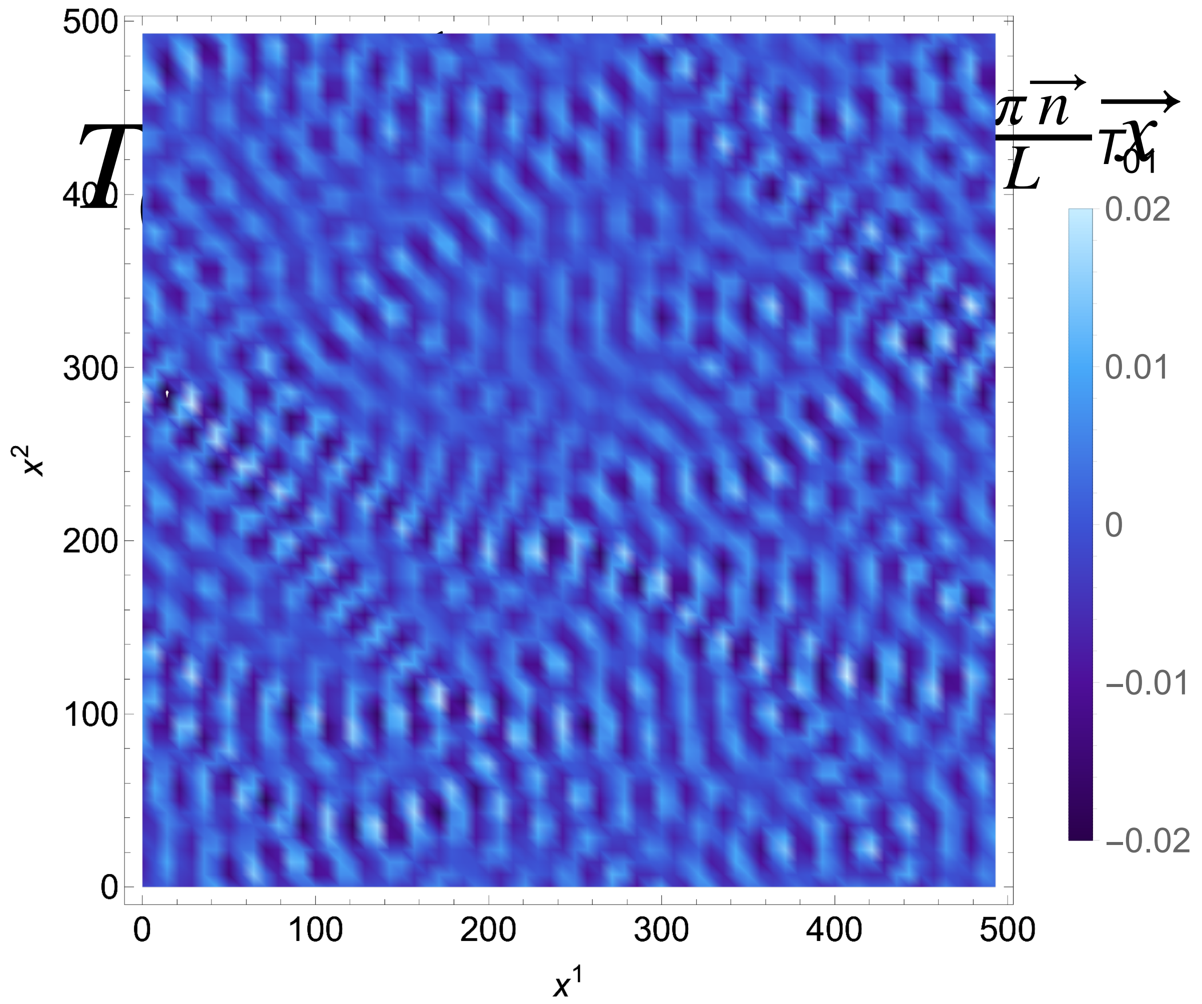

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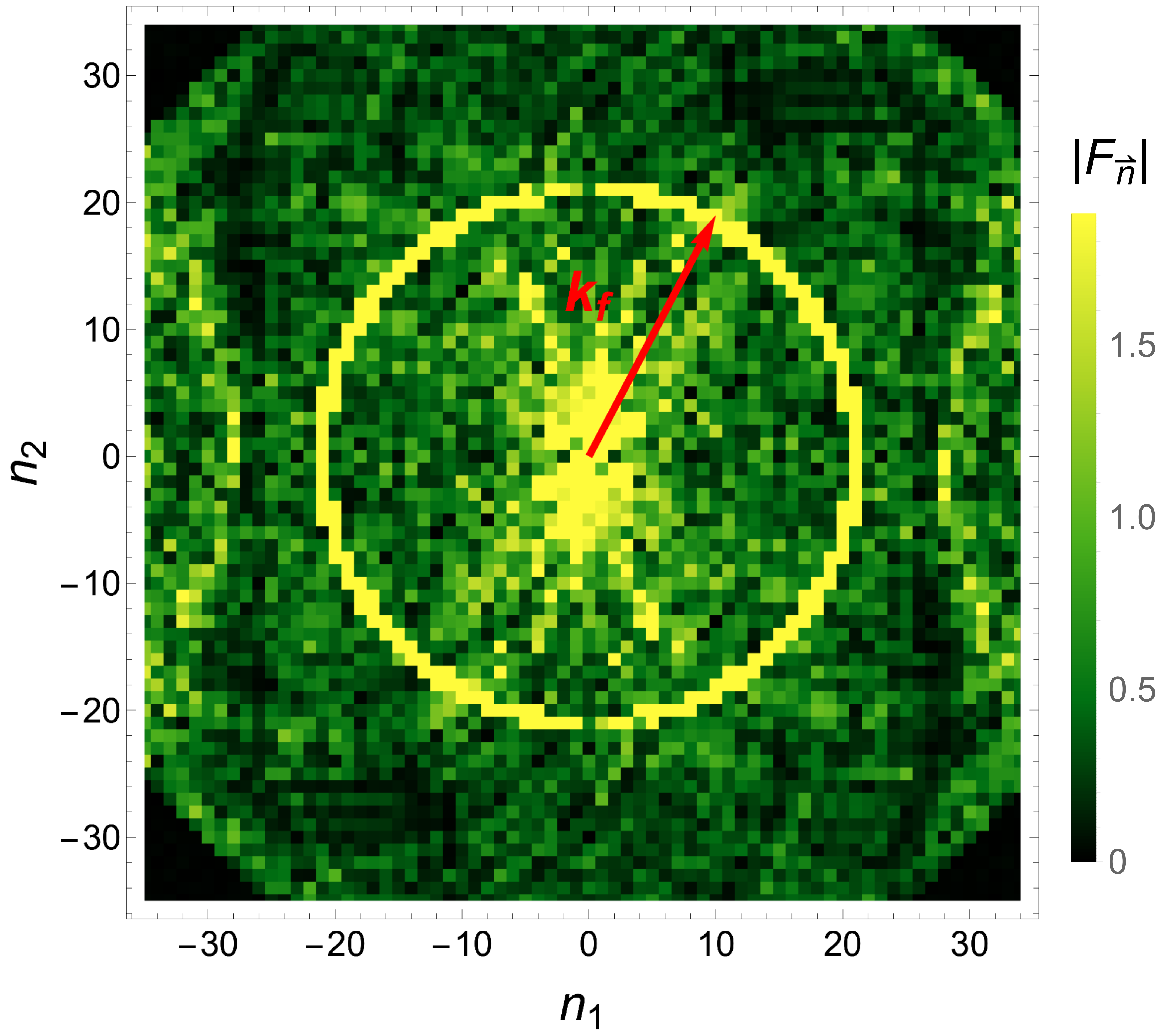

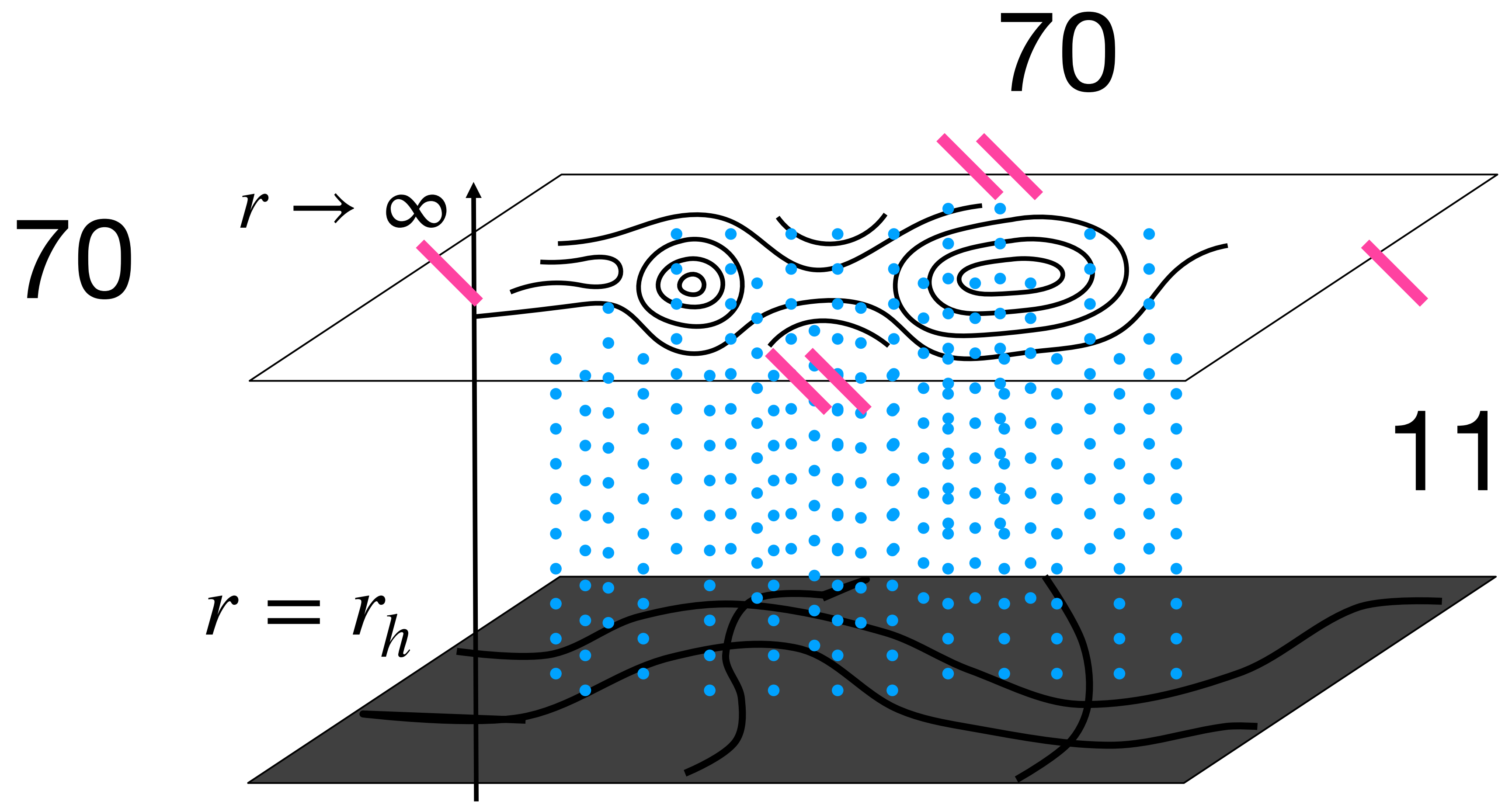

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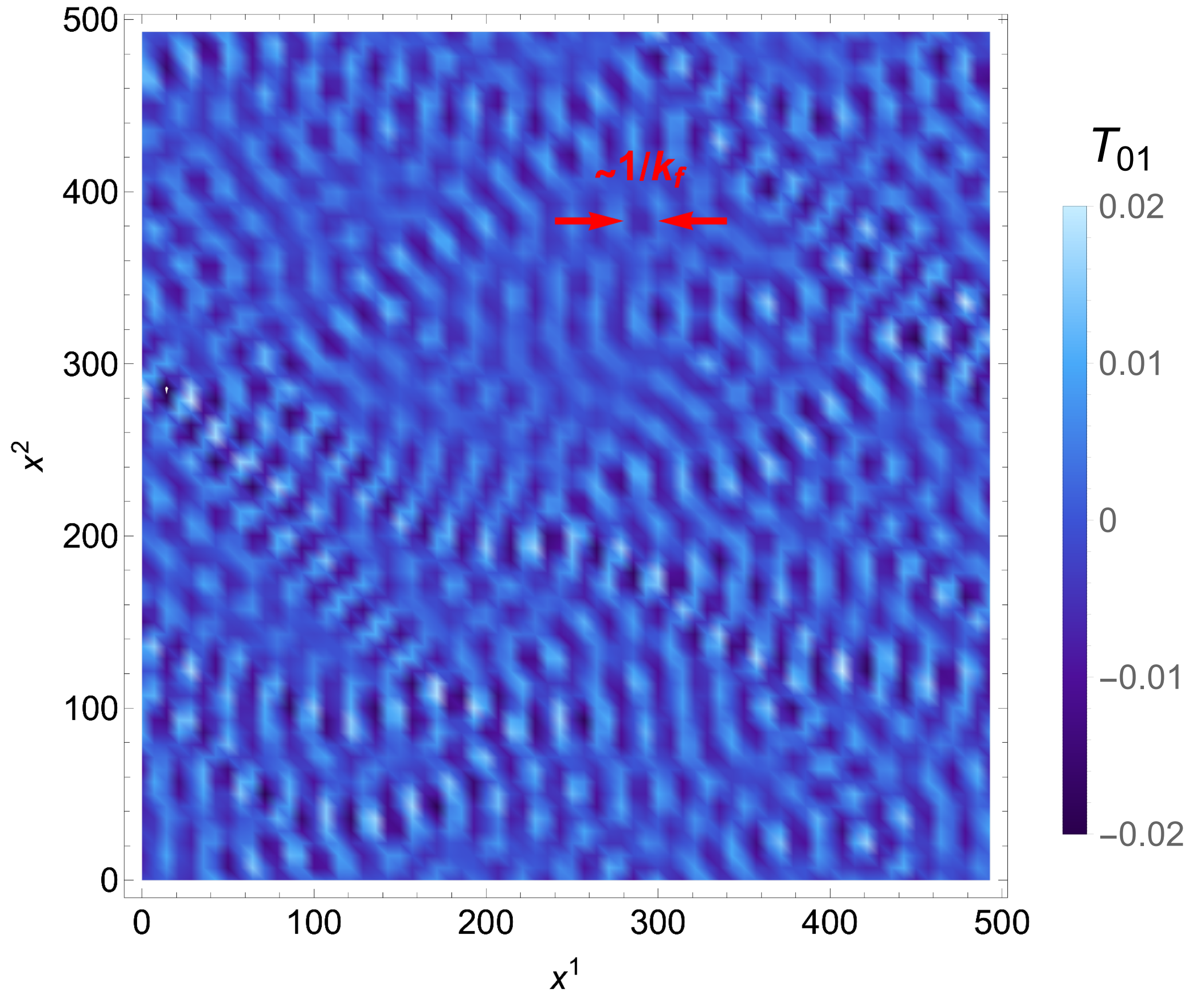

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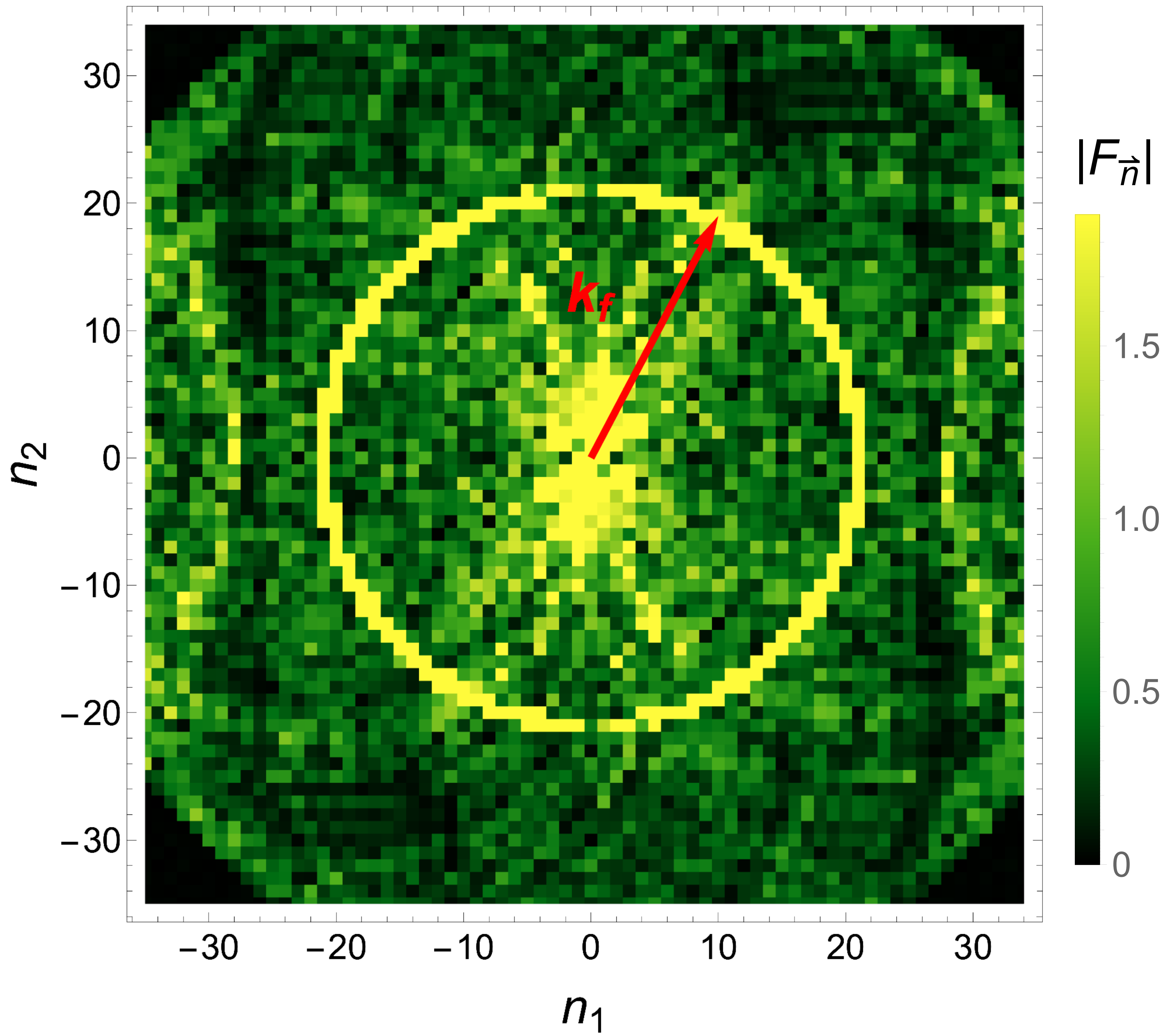

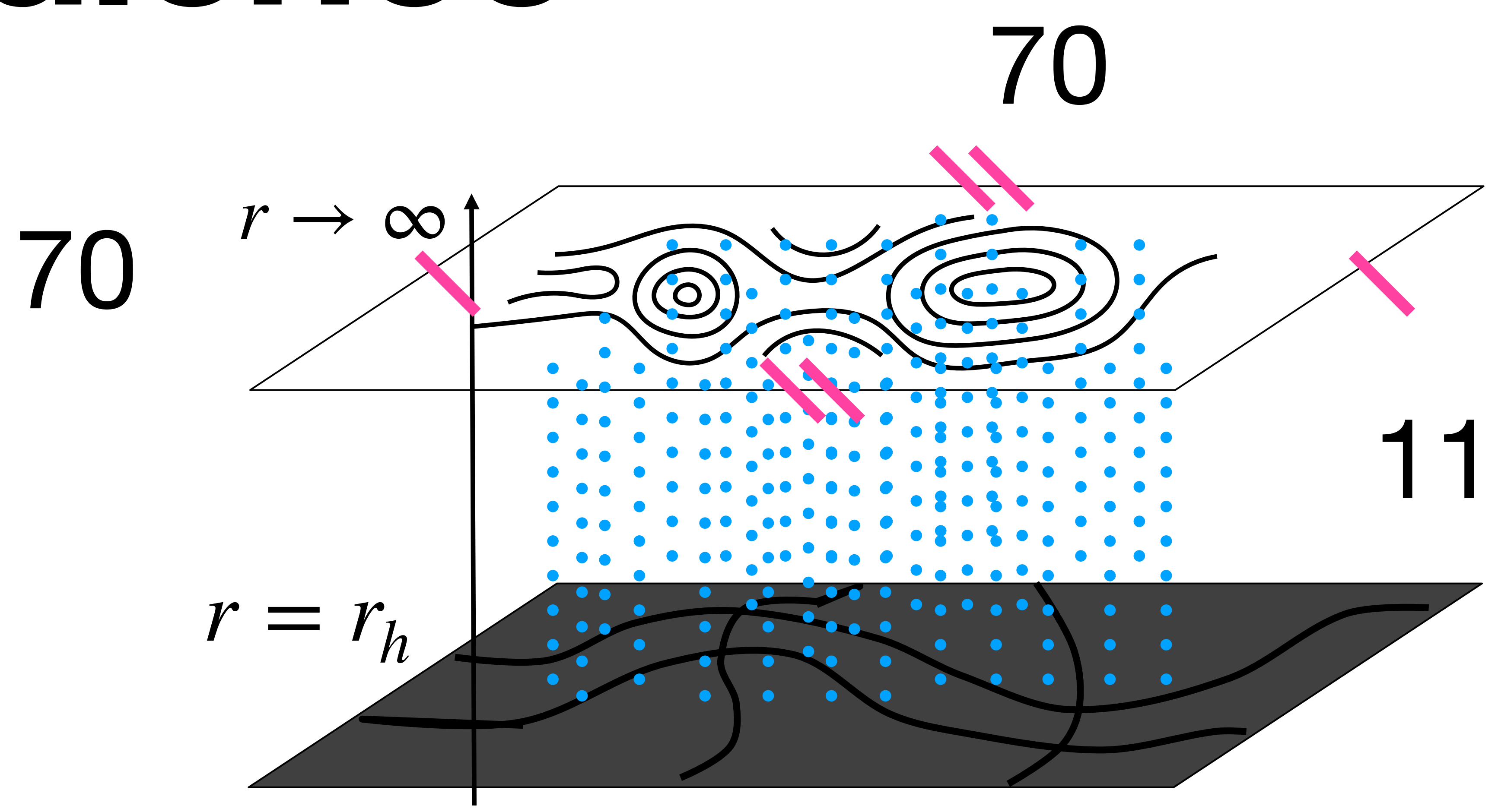

$$\hat{F}(|\vec{n}|) = \int F_{\vec{n}} |\vec{n}| d\theta$$