Three-Nucleon Forces with Symmetry Preserving Regulator

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Outline

- Path-integral approach for derivation of nuclear forces
- Symmetry preserving regularization
- Status report on construction of 3N interactions

Path-Integral Framework for Derivation of Nuclear Forces

HK, Epelbaum, arXiv:2311.10893

Why a new Framework?

Difficulties in formulation of regularized chiral EFT

- Regularization should preserve chiral and gauge symmetries

- Regularization should not affect long-range pion physics

Pion-propagator in Euclidean space: $q^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$

$$\frac{1}{q^2 + M_\pi^2} \to \frac{\exp\left(-\frac{q^2+M_\pi^2}{\Lambda^2}\right)}{q^2 + M_\pi^2} = \frac{1}{q^2 + M_\pi^2} - \frac{1}{\Lambda^2} + \frac{q^2 + M_\pi^2}{2\Lambda^4} + \ldots$$

all 1/Λ-corrections are short-range interactions

q_0 - dependence in exponential requires second and higher order time-derivatives in pion field in the chiral Lagrangian

→ Canonical quantization of the regularized theory becomes difficult (Ostrogradski - approach, Constrains, …)

Canonical vs Path-Integral Quantization

Canonical Quantization of QFT

Hamiltonian & Hilbert space

Creation/annihilation operators

Time-ordered perturbation theory

↔

Path-Integral Quantization of QFT

Lagrangian & action

Summation over all classical paths

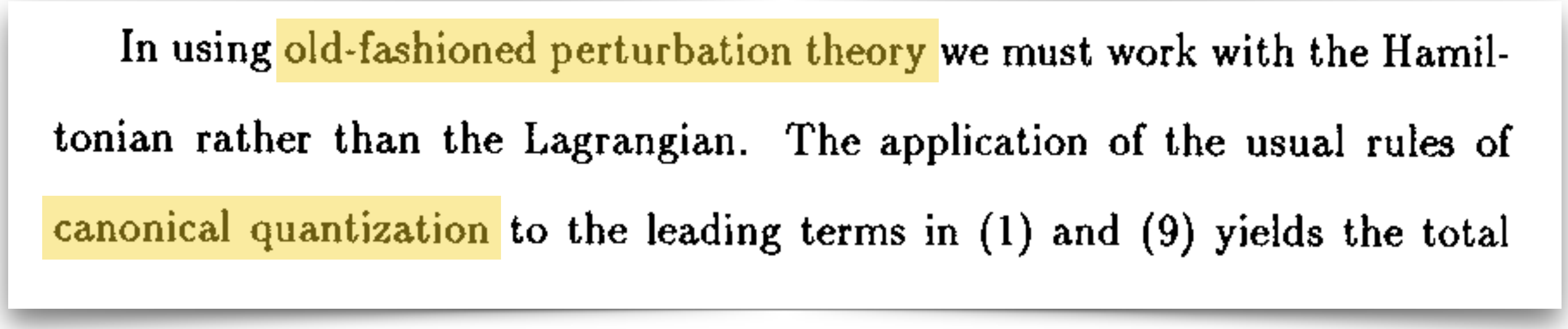
Loop expansion & Feynman rules

- Path-Integral approach is a natural choice in pionic and single-nucleon sector

Gasser, Leutwyler, Annals Phys. 158 (1984) 142;
Bernard, Kaiser, Kambor, Meißner, Nucl. Phys. B 388 (1992) 315

- In two - and more - nucleon sector Weinberg used canonical quantization language

Weinberg Nucl. Phys. B 362 (1991) 3



In using old-fashioned perturbation theory we must work with the Hamiltonian rather than the Lagrangian. The application of the usual rules of canonical quantization to the leading terms in (1) and (9) yields the total

Can we choose a formulation where we can work with the Lagrangian?

Path-Integral over Nucleons and Pions

We start with generating functional:

$$Z[\eta^\dagger,\eta] = \int [DN^\dagger][DN][D\pi]\exp\left(i \int d^4x \big(\mathscr{L} + \eta^\dagger(x)N(x) + N^\dagger(x)\eta(x) \big) \right)$$

Yukawa toy-model:

$$\mathscr{L} = N^\dagger \left(i\frac{\partial}{\partial x_0} + \frac{\vec{\nabla}^2}{2m} + \frac{g}{2F}\vec{\sigma}\cdot\vec{\nabla}\pi\cdot\tau \right) N + \frac{1}{2}\big(\partial_\mu\pi\cdot\partial^\mu\pi - M^2\pi^2\big)$$

- Perform a Gaussian path-integral over the pion fields

$$Z[\eta^\dagger,\eta] = \int [DN^\dagger][DN]\exp\left(i\,S_N + i \int d^4x \big(\eta^\dagger(x)N(x) + N^\dagger(x)\eta(x) \big) \right)$$

$$S_N = \int d^4x\, N^\dagger(x)\left(i\frac{\partial}{\partial x_0} + \frac{\vec{\nabla}^2}{2m} \right) N(x) \; - \; V_{NN} \;\leftarrow$$

Non-instant one-pion-exchange interaction

$$V_{NN} = -\frac{g^2}{8F^2}\int d^4x\, d^4y\, \vec{\nabla}_x\cdot\big[N^\dagger(x)\vec{\sigma}\tau\big]N(x)\,\Delta_F(x-y)\,\vec{\nabla}_y\cdot\big[N^\dagger(y)\vec{\sigma}\tau\big]N(y)$$

with non-instant pion propagator: $\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4}\frac{e^{-i\,q\cdot x}}{q^2 - M^2 + i\,\epsilon}$

Instant Interactions from Path-Integral

To transform V_{NN} into an instant form we rewrite a pion propagator

$$\frac{1}{q_0^2-\omega_q^2} = -\frac{1}{\omega_q^2} + \frac{1}{q_0^2-\omega_q^2} + \frac{1}{\omega_q^2} = -\frac{1}{\omega_q^2} + q_0^2 \frac{1}{\omega_q^2}\frac{1}{q_0^2-\omega_q^2}, \qquad \omega_q = \sqrt{\vec{q}^{\,2}+M^2}$$

In coordinate space this corresponds to $\Delta_F(x) = \Delta_S(x) - \frac{\partial^2}{\partial x_0^2}\Delta_{FS}(x)$ with

$$\Delta_S(x) = -\int \frac{d^4q}{(2\pi)^4}\frac{e^{-i\,q\cdot x}}{\omega_q^2} = -\,\delta(x_0)\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\,\vec{q}\cdot\vec{x}}}{\omega_q^2}, \qquad \Delta_{FS}(x) = \int \frac{d^4q}{(2\pi)^4}\frac{e^{-i\,q\cdot x}}{\omega_q^2(q_0^2-\omega_q^2)}$$

- The decomposition $\Delta_F(x) = \Delta_S(x) - \frac{\partial^2}{\partial x_0^2}\Delta_{FS}(x)$ can be generalized

$G(x) = \int \frac{d^4q}{(2\pi)^4} e^{-i\,q\cdot x}\tilde{G}(q_0^2, q^2)$ and $\tilde{G}(q_0^2, q^2)$ is differentiable at $q_0 = 0$

Defining $G_S(x) = \int \frac{d^4q}{(2\pi)^4} e^{-i\,q\cdot x}\tilde{G}(0,q^2)$ and $G_{FS}(x) = \int \frac{d^4q}{(2\pi)^4} e^{-i\,q\cdot x}\frac{\tilde{G}(q_0^2, q^2) - \tilde{G}(0,q^2)}{q_0^2}$

$$\rightarrow \quad \boxed{G(x) = G_S(x) - \frac{\partial^2}{\partial x_0^2} G_{FS}(x)}$$

Instant Interactions from Path-Integral

Perform an instant decomposition of the pion propagator $\Delta_F(x) = \Delta_S(x) - \frac{\partial^2}{\partial x_0^2}\Delta_{FS}(x)$

$$V_{NN} = -\frac{g^2}{8F^2}\int d^4x\, d^4y\, \vec{\nabla}_x \cdot \left[N^\dagger(x)\vec{\sigma}\tau\right]N(x)\, \Delta_F(x-y)\, \vec{\nabla}_y \cdot \left[N^\dagger(y)\vec{\sigma}\tau\right]N(y)$$

$$\rightarrow \quad V_{NN} = V_{OPE} + V_{FS}$$

$$V_{OPE} = -\frac{g^2}{8F^2}\int d^4x\, d^4y\, \vec{\nabla}_x \cdot \left[N^\dagger(x)\vec{\sigma}\tau\right]N(x)\, \Delta_S(x-y)\, \vec{\nabla}_y \cdot \left[N^\dagger(y)\vec{\sigma}\tau\right]N(y)$$ is instant

$$V_{FS} = \frac{g^2}{8F^2}\int d^4x\, d^4y\, \vec{\nabla}_x \cdot \left[N^\dagger(x)\vec{\sigma}\tau\right]N(x)\, \frac{\partial^2}{\partial x_0^2}\Delta_{FS}(x-y)\, \vec{\nabla}_y \cdot \left[N^\dagger(y)\vec{\sigma}\tau\right]N(y)$$ is non-instant

V_{FS} is time-derivative dependent and thus can be eliminated by a non-polynomial field redefinition

$$N(x) \rightarrow N'(x) = N(x) + i\frac{g^2}{8F^2}\int d^4y\, [\vec{\sigma}\tau N(x)] \cdot [\, \vec{\nabla}_x \frac{\partial}{\partial x_0}\Delta_{FS}(x-y)]\, \vec{\nabla}_y \cdot [N^\dagger(y)\vec{\sigma}\tau N(y)]$$

$$N^\dagger(x) \rightarrow N'^\dagger(x) = N^\dagger(x) - i\frac{g^2}{8F^2}\int d^4y\, \vec{\nabla}_y \cdot [N^\dagger(y)\vec{\sigma}\tau N(y)][\, \vec{\nabla}_y \frac{\partial}{\partial y_0}\Delta_{FS}(y-x)] \cdot [N^\dagger(x)\vec{\sigma}\tau]$$

Instant Interactions from Path-Integral

Non-local field transformations remove time-derivative dependent two-nucleon interactions but generate time-derivative dependent three-nucleon interactions.

These contributions can be eliminated by similar field transformations

$$Z[\eta^\dagger,\eta] = \int [DN'^\dagger][DN'] \det\left(\frac{\delta(N'^\dagger,N')}{\delta(N^\dagger,N)}\right)\exp\left(i\,S_{N(N'^\dagger,N')} + i\int d^4x\big(\eta^\dagger(x)N(N'^\dagger,N')(x) + N(N'^\dagger,N')^\dagger(x)\eta(x)\big)\right)$$

$$\simeq \int [DN'^\dagger][DN'] \det\left(\frac{\delta(N'^\dagger,N')}{\delta(N^\dagger,N)}\right)\exp\left(i\,S_{N(N'^\dagger,N')} + i\int d^4x\big(\eta^\dagger(x)N'(x) + N'^\dagger(x)\eta(x)\big)\right)$$

Equivalence theorem: nucleon pole-structure is unaffected by the field-transf.

$$S_{N(N'^\dagger,N')} = \int d^4x\, N'^\dagger(x)\left(i\frac{\partial}{\partial x_0} + \frac{\vec{\nabla}^2}{2m}\right)N'(x) \;-\; V_{OPE} + \mathcal{O}(g^4)$$

$$V_{OPE} = -\frac{g^2}{8F^2}\int d^4x\, d^4y\, \vec{\nabla}_x\cdot\big[N'^\dagger(x)\vec{\sigma}\tau\big]N'(x)\,\Delta_S(x-y)\,\vec{\nabla}_y\cdot\big[N'^\dagger(y)\vec{\sigma}\tau\big]N'(y)$$

Instant one-pion-exchange interaction

Generalization to Chiral EFT

We start with generating functional:

$$Z[\eta^\dagger, \eta] = \int [DN^\dagger][DN][D\pi] \exp\left(i \int d^4x \big(\mathscr{L}_\pi + \mathscr{L}_{\pi N} + \mathscr{L}_{NN} + \mathscr{L}_{NNN} + \eta^\dagger(x) N(x) + N^\dagger(x) \eta(x) \big) \right)$$

- Integrate over pion fields via loop-expansion of the action

  → expansion of the action around the classical pion solution

- Perform instant decomposition of the remaining interactions between nucleons
- Perform nucleon-field redefinitions to eliminate non-instant part of the interaction
- Calculate functional determinant to get one-loop corrections to few-nucleon forces

Connection to Unitary Transformations

Previous derivation of nuclear forces was based on unitary transformation technique

Field transformations (FT) within path-integral approach $\overset{?}{\longleftrightarrow}$ Unitary transformations (UT) within canonical quantization approach

- Interactions generated by FT have always a form of heavy-baryon like tree-level or 4-dim loop-integrals

- Interactions generated by UT can be matched by 4-dim loop-integrals, *only* if some unitary phases are fixed

  → UT technique is more flexible

In practical calculation we do not want to explore the flexibility of UT in constructing non-renormalizable nuclear forces

- FT which don't generate interactions with time-derivatives describe off-shell ambiguities

- Allows to study unitary ambiguities of e.g. relativistic corrections

UT & FT path-integral approach lead to the same chiral EFT nuclear forces up to N^{4}LO

Fazit: Path-integral formulation of nuclear forces is as powerful as UT technique, however it allows consideration of a wider class of theories

Symmetry Preserving Regulator

HK, Epelbaum, arXiv:2312.13932

Gradient-Flow Equation (GFE)

Yang-Mills gradient flow in QCD: **Lüscher, JHEP 04 (2013) 123**

$$\partial_\tau B_\mu = D_\nu G_{\nu\mu} \quad \text{with} \quad B_\mu|_{\tau=0} = A_\mu \quad \& \quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

B_μ is a regularized gluon field

- Apply this idea to ChPT: **HK, Epelbaum, arXiv:2312.13932**

(Proposed in various talks by D. Kaplan for nuclear forces)

Introduce a smoothed pion field W with $W|_{\tau=0} = U$ satisfying GFE

$$\partial_\tau W = i\, w\, \text{EOM}(\tau)\, w \quad \text{with} \quad w = \sqrt{W} \quad \text{and} \quad \text{EOM}(\tau) = [D_\mu, w_\mu] + \frac{i}{2}\chi_- - \frac{i}{4}\text{Tr}(\chi_-)$$

$$w_\mu = i(w^\dagger(\partial_\mu - i\, r_\mu)w - w(\partial_\mu - i\, l_\mu)w^\dagger), \quad \chi_- = w^\dagger \chi w^\dagger - w\chi^\dagger w, \quad \chi = 2B(s + ip)$$

Note: The shape of regularization is dictated by the choice of the right-hand side of GFE

- Our choice is motivated by a Gaussian regularization of one-pion-exchange in NN

Properties under Chiral Transformation

Replace all pion fields in pion-nucleon Lagrangians $\mathscr{L}_{\pi N}^{(1)}, \ldots, \mathscr{L}_{\pi N}^{(4)}$: $U \to W$

$$\mathscr{L}_{\pi N}^{(1)} = N^\dagger \Big(D^0 + g\, u \cdot S \Big) N \to N^\dagger \Big(D_w^0 + g\, w \cdot S \Big) N$$

Chiral transformation: by induction, one can show

$$U \to R U L^\dagger \quad \longrightarrow \quad W \to R W L^\dagger$$

- Regularized pion fields transform under τ - independent transformations

- Nucleon fields transform in τ - dependent way

$$N \to K N, \quad K = \sqrt{L U^\dagger R^\dagger} R \sqrt{U} \quad \longrightarrow \quad N \to K_\tau N, \quad K_\tau = \sqrt{L W^\dagger R^\dagger} R \sqrt{W}$$

Gradient-Flow Equation

Analytic solution is possible of $1/F$ - expanded gradient flow equation:

$$W = 1 + i\tau \cdot \phi(1 - \alpha\phi^2) - \frac{\phi^2}{2}\left[1 + \left(\frac{1}{4} - 2\alpha\right)\phi^2\right] + \mathcal{O}(\phi^5), \quad \phi_b = \sum_{n=0}^{\infty} \frac{1}{F^n}\phi_b^{(n)}$$

In the absence of external sources we have

$$\left[\partial_\tau - (\partial_\mu^x \partial_\mu^x - M^2)\right]\phi_b^{(1)}(x,\tau) = 0, \quad \phi_b^{(1)}(x,0) = \pi_b(x)$$

$$\left[\partial_\tau - (\partial_\mu^x \partial_\mu^x - M^2)\right]\phi_b^{(3)}(x,\tau) = (1-2\alpha)\partial_\mu\phi^{(1)} \cdot \partial_\mu\phi^{(1)}\phi_b^{(1)} - 4\alpha\partial_\mu\phi^{(1)} \cdot \phi^{(1)}\partial_\mu\phi_b^{(1)} + \frac{M^2}{2}(1-4\alpha)\phi^{(1)} \cdot \phi^{(1)}\phi_b^{(1)}, \quad \phi_b^{(3)}(x,0) = 0$$

Iterative solution in momentum space: $\tilde{\phi}^{(n)}(q,\tau) = \int d^4x\, e^{iq\cdot x}\phi_b^{(n)}(x,\tau)$

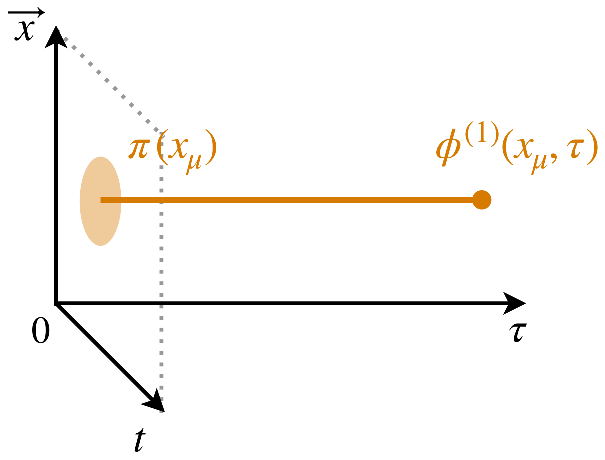
$$\tilde{\phi}_b^{(1)}(q) = e^{-\tau(q^2+M^2)}\tilde{\pi}_b(q)$$

$$\tilde{\phi}_b^{(3)}(q) = \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}(2\pi)^4\delta(q - q_1 - q_2 - q_3)\int_0^\tau ds\, e^{-(\tau-s)(q^2+M^2)}e^{-s\sum_{j=1}^3(q_j^2+M^2)}$$
$$\times\left[4\alpha\, q_1 \cdot q_3 - (1-2\alpha)q_1 \cdot q_2 + \frac{M^2}{2}(1-4\alpha)\right]\tilde{\pi}(q_1) \cdot \tilde{\pi}(q_2)\tilde{\pi}_b(q_3)$$

Integration over momenta of pion fields with Gaussian prefactor introduces smearing

Iterative solution in Coordinate Space

$\phi(x_\mu, \tau)$ =



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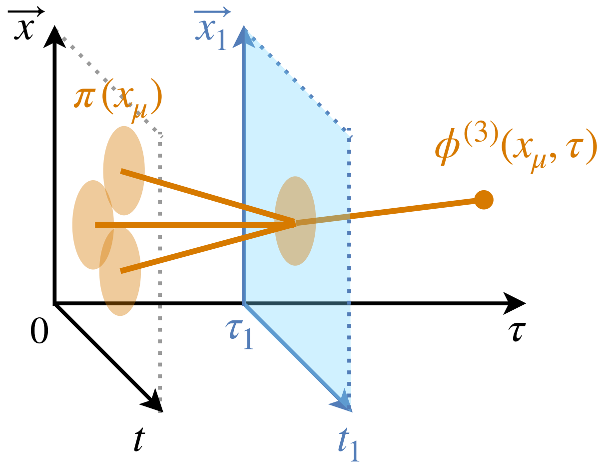


+ …

[integrated over $\vec{x_1}$, t_1, τ_1]

Light-shaded area visualizes smearing in Euclidean space of size $\sim \sqrt{2\tau}$

Solid line stands for Green-function:

$$\left[\partial_\tau - (\partial^x_\mu \partial^x_\mu - M^2)\right] G(x-y, \tau-s) = \delta(x-y)\delta(\tau-s)$$

$$G(x,\tau) = \theta(\tau) \int \frac{d^4q}{(2\pi)^4} e^{-\tau(q^2+M^2)} e^{-i\,q\cdot x}$$

$$\phi_b^{(1)}(x,\tau) = \int d^4y\, G(x-y,\tau)\pi_b(y)$$

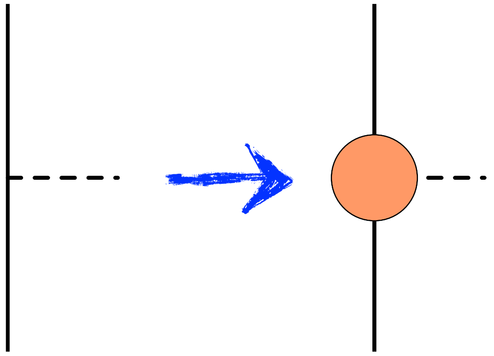
$$\phi_b^{(3)}(x,\tau) = \int_0^\tau ds \int d^4y\, G(x-y,\tau-s)\Big[(1-2\alpha)\partial_\mu\phi^{(1)}(y,s)\cdot\partial_\mu\phi^{(1)}(y,s)\phi_b^{(1)}(y,s)$$
$$-\,4\alpha\,\partial_\mu\phi^{(1)}(y,s)\cdot\phi^{(1)}(y,s)\partial_\mu\phi_b^{(1)}(y,s) + \frac{M^2}{2}\phi^{(1)}(y,s)\cdot\phi^{(1)}(y,s)\phi_b^{(1)}(y,s)\Big]$$

Regularization for Nuclear Forces

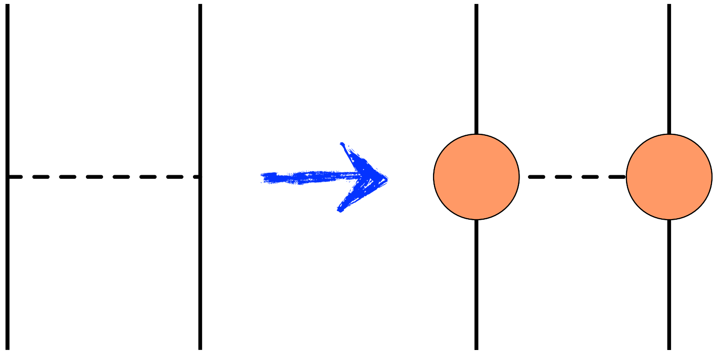
To regularize long-range part of the nuclear forces and currents

- Leave pionic Lagrangians $\mathscr{L}_{\pi}^{(2)}$ & $\mathscr{L}_{\pi}^{(4)}$ unregularized (essential)
- Replace all pion fields in pion-nucleon Lagrangians $\mathscr{L}_{\pi N}^{(1)}, \ldots, \mathscr{L}_{\pi N}^{(4)}$: $U \to W$

$$\mathscr{L}_{\pi N}^{(1)} = N^\dagger\left(D^0 + g\, u\cdot S\right)N \to N^\dagger\left(D^0_w + g\, w\cdot S\right)N$$



$\sim e^{-\tau(q^2+M^2)}$



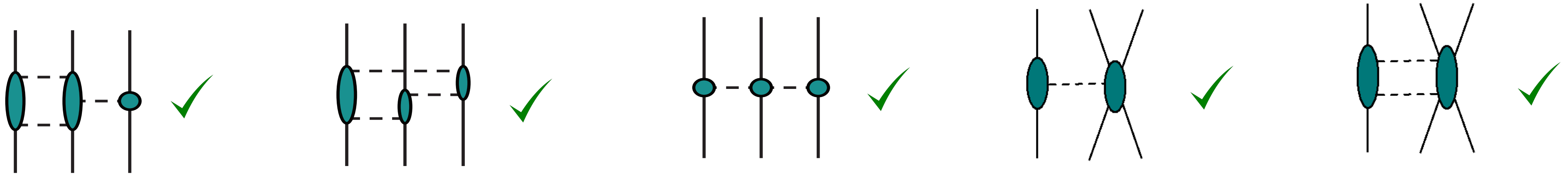
$\sim e^{-2\tau(q^2+M^2)}\dfrac{1}{q^2+M^2}$

For $\tau = \dfrac{1}{2\Lambda^2}$ this regulator reproduces SMS regularization of OPE

Status Report on 3NF

Status Report on 3N at N³LO

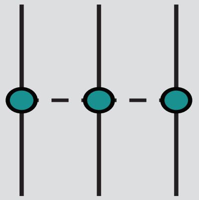
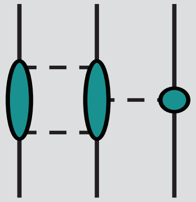
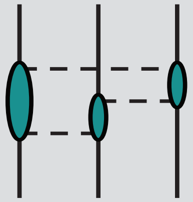
- We calculated all long- and short-range contributions to 3NF & 4NF at N³LO



3NF’s are given in terms of integrals over Schwinger parameters

$$V_{3N}^{2\pi-1\pi} = \tau_1 \cdot \tau_2 \times \tau_3 \vec{q}_1 \cdot \vec{\sigma}_1 \times \vec{\sigma}_2 \vec{q}_3 \cdot \vec{\sigma}_3 \frac{e^{-\frac{q_3^2+M_\pi^2}{\Lambda^2}}}{q_3^2+M_\pi^2} \left(- \frac{g_A^4}{F_\pi^6} \frac{q_1}{2048\pi} \int_0^\infty d\lambda \, \mathrm{erfi}\left(\frac{q_1 \lambda}{2\Lambda\sqrt{2+\lambda}} \right) \frac{\exp\left(- \frac{q_1^2+4M_\pi^2}{4\Lambda^2}(2+\lambda) \right)}{2+\lambda} + \ldots \right) + \ldots$$

Dimension of integrals over Schwinger parameters depends on topology

Space			
Momentum	2	1	3
Coordinate	4	1	0

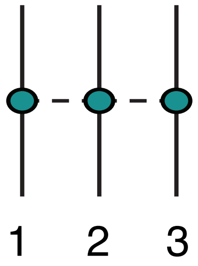
Subtraction Scheme

Choice of the short-range scheme

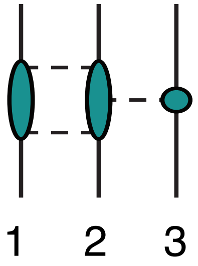
- NN case: local part of NN force vanishes if distance between nucleons vanishes

→ leads to natural size of LECs

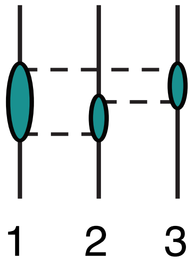
- 3N case: vanishing of the local part of 3NF is topology dependent



$= 0$ if $r_{12} = 0$ or $r_{23} = 0$

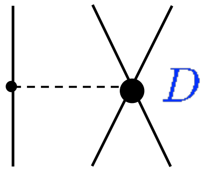


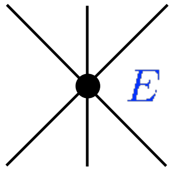
$= 0$ if $r_{12} = 0$



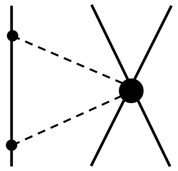
$= 0$ if $r_{12} = r_{23} = 0$

Can be achieved by adjustment of D- and E-like terms:





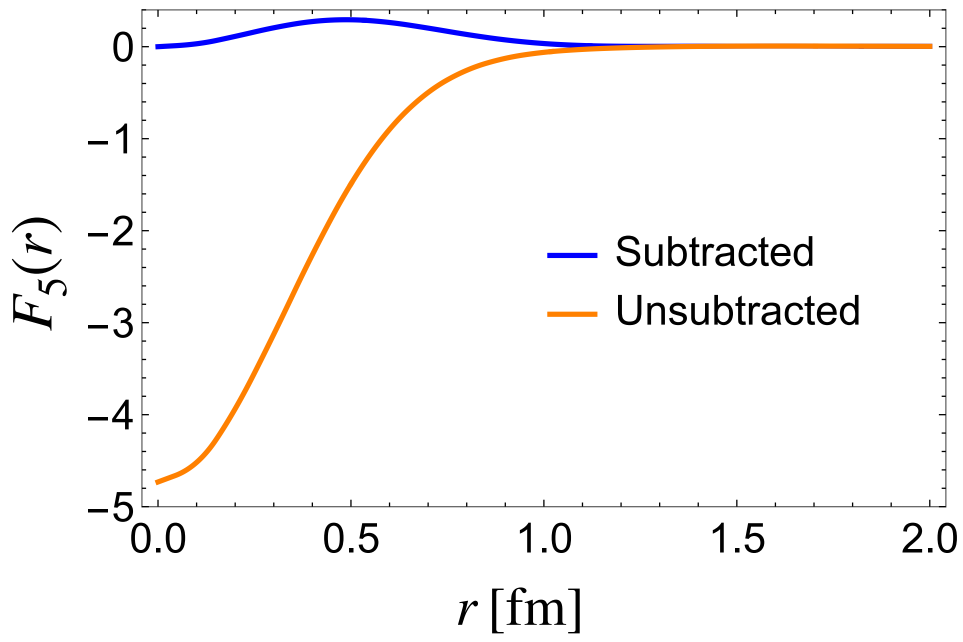
Vanishing of 3NF for any $r_{ij} = 0$ would require inclusion of two-pion-contact terms

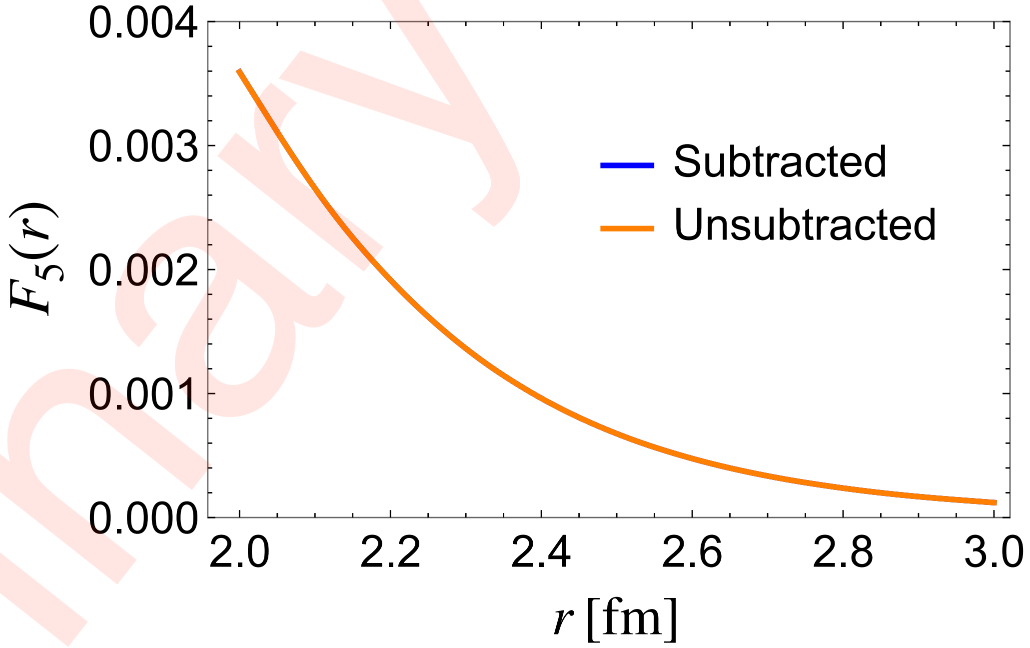


Appear first at N⁵LO and are expected to be small

Selected Profile Functions

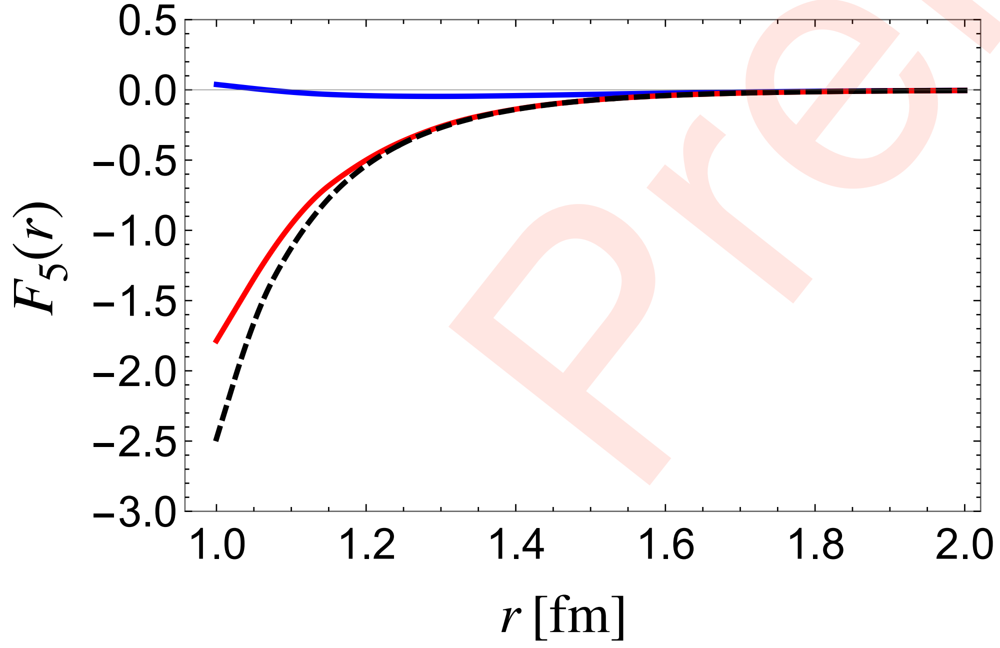
$V_{3N}^{\text{ring}} = F_1(r_{12}, r_{23}, r_{13}) + \ldots + \tau_2 \cdot \tau_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2 F_5(r_{12}, r_{23}, r_{13}) + \ldots$ $\quad F_5(r) = F_5(r, r, r)$ [MeV]

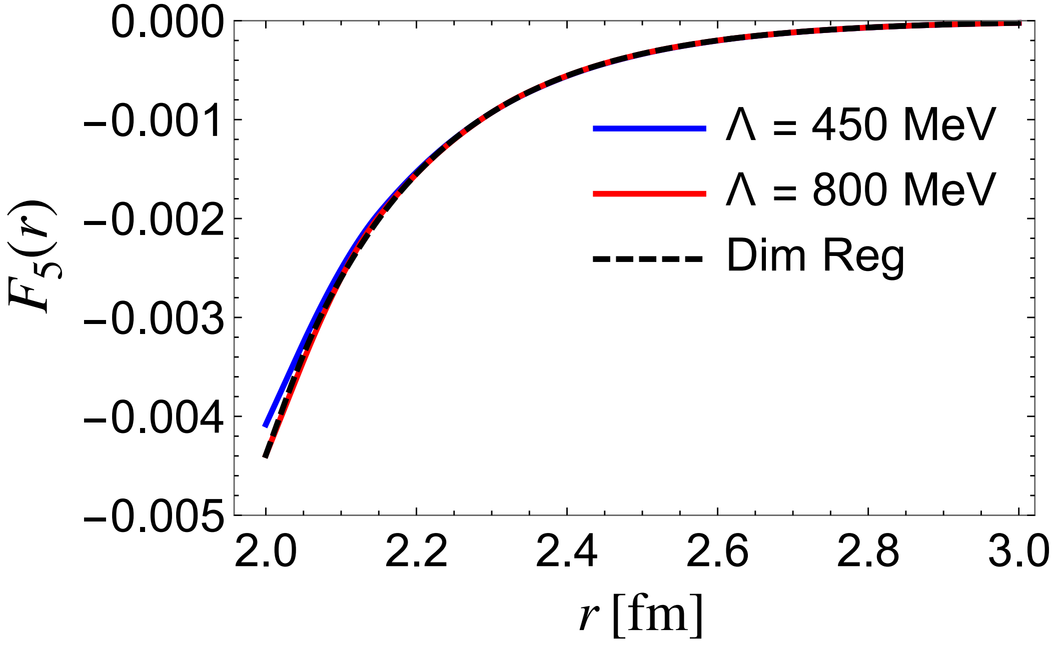




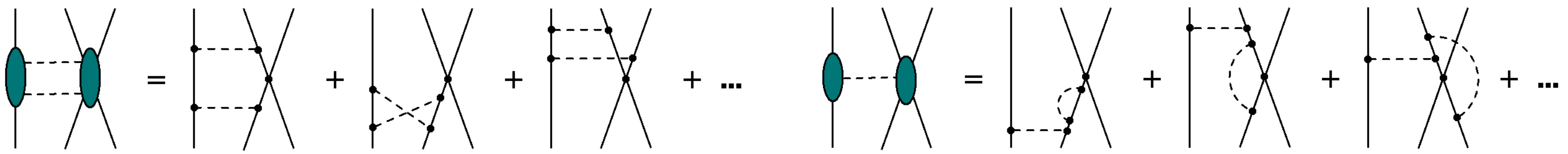
By construction: subtracted & unsubtracted forces differ in the short-range region

At $\Lambda \to \infty$ regularized 3NF reproduce dim. reg. results from **Bernard et al. PRC77 (08)**





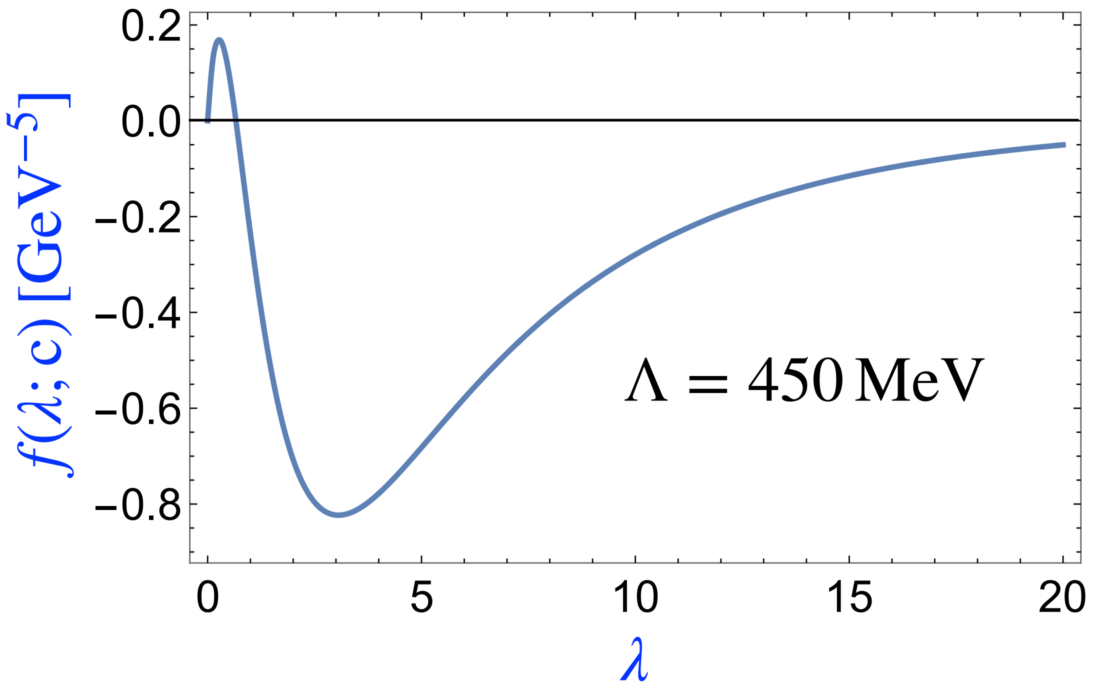
Short-Range Part on 3NF at N3LO



- Non-locality introduces additional momenta
- To get a finite 3NF in $\Lambda \to \infty$ limit we have to perform 5 additional field-transformations which include second power of the pion propagator

→ more extensive calculation

Short-range parts are given in terms of 1-dim integrals over Schwinger parameters



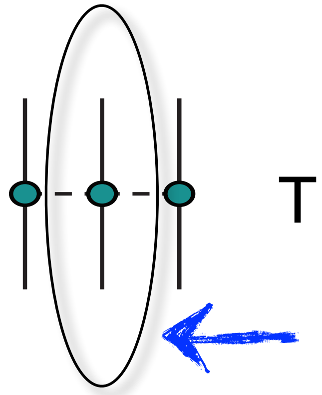
Selected structure & configuration c:

$$\hat{q}_1 \cdot \vec{\sigma}_2 \hat{q}_1 \cdot \vec{\sigma}_3 \tau_1 \cdot \tau_3\, C_S \int_0^\infty d\lambda\, f(\lambda; \mathrm{c})$$

c includes momenta in MeV & cosines of angles:

q_1	k_1	q_{23}	k_{23}	$\hat{q}_1 \cdot \hat{k}_1$	$\hat{q}_1 \cdot \hat{q}_{23}$	$\hat{q}_1 \cdot \hat{k}_{23}$	$\hat{k}_1 \cdot \hat{q}_{23}$	$\hat{k}_1 \cdot \hat{k}_{23}$	$\hat{q}_{23} \cdot \hat{k}_{23}$
1	$\frac{1}{2}$	3	2	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{7}$	$-\frac{1}{9}$	$\frac{1}{8}$

Homework



- TPE topology includes pion-nucleon amplitude as a subprocess

  Pion-nucleon amplitude with gradient-flow regulator depends on c_i's

  Fit c_i's to pion-nucleon sub-threshold coefficients which are determined from Roy-Steiner equation

  Calculation of pion-nucleon scattering with gradient-flow regulator required

- Partial wave decomposition (PWD): K. Hebeler, A. Nogga & R. Skibinski

  PWD is computationally more expensive, due to higher dimension of integrals over Schwinger parameters

Summary

- Path-integral approach for derivation of nuclear forces
- Gradient flow regularization preserves chiral symmetry
- Long- & short-range part of 3NF at N^3LO is calculated

Outlook

- Pion-nucleon scattering with gradient-flow regulator
- Partial wave decomposition
- Symmetry preserving regularized nuclear currents