TRIUMF

Standard Model corrections to Fermi transitions in light nuclei

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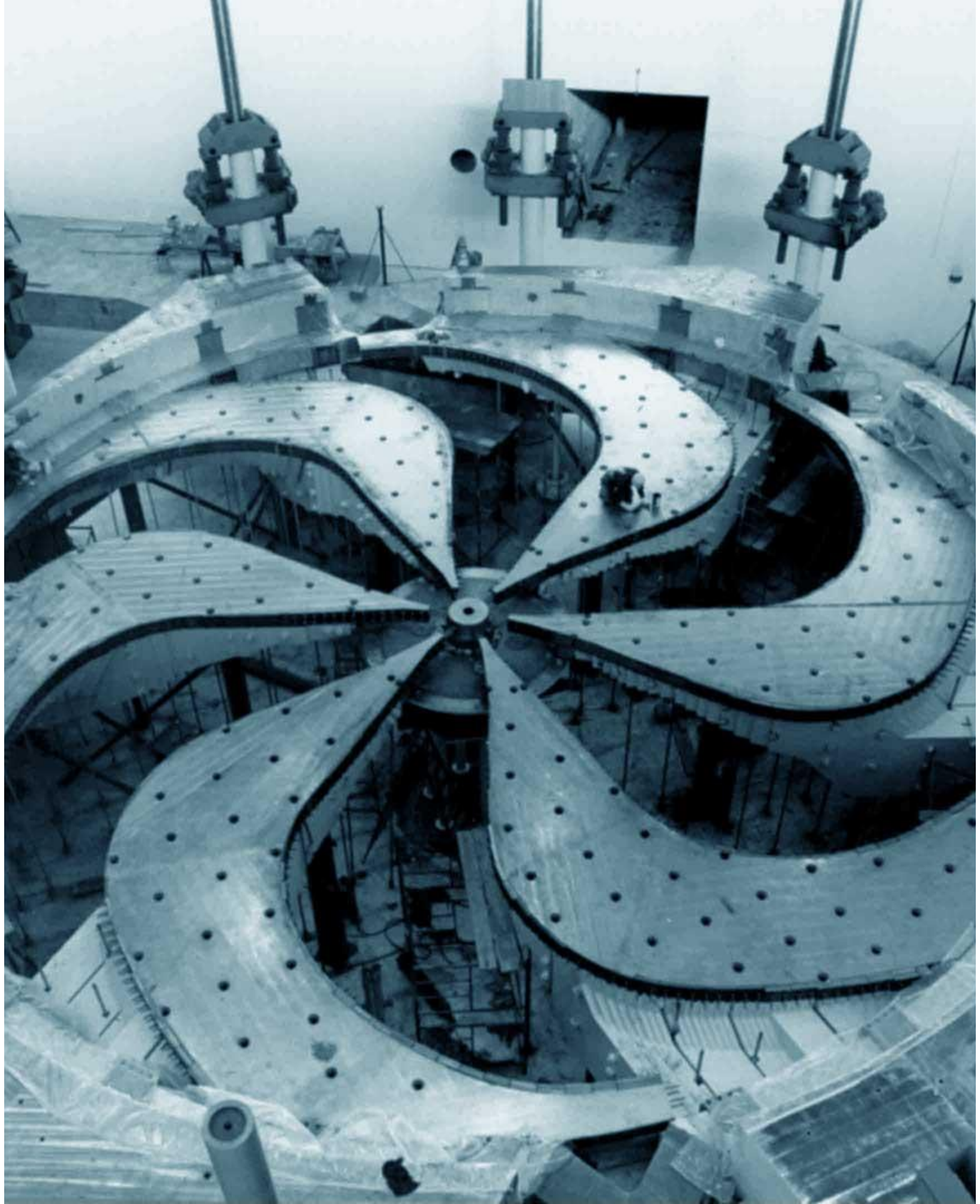
FRIB

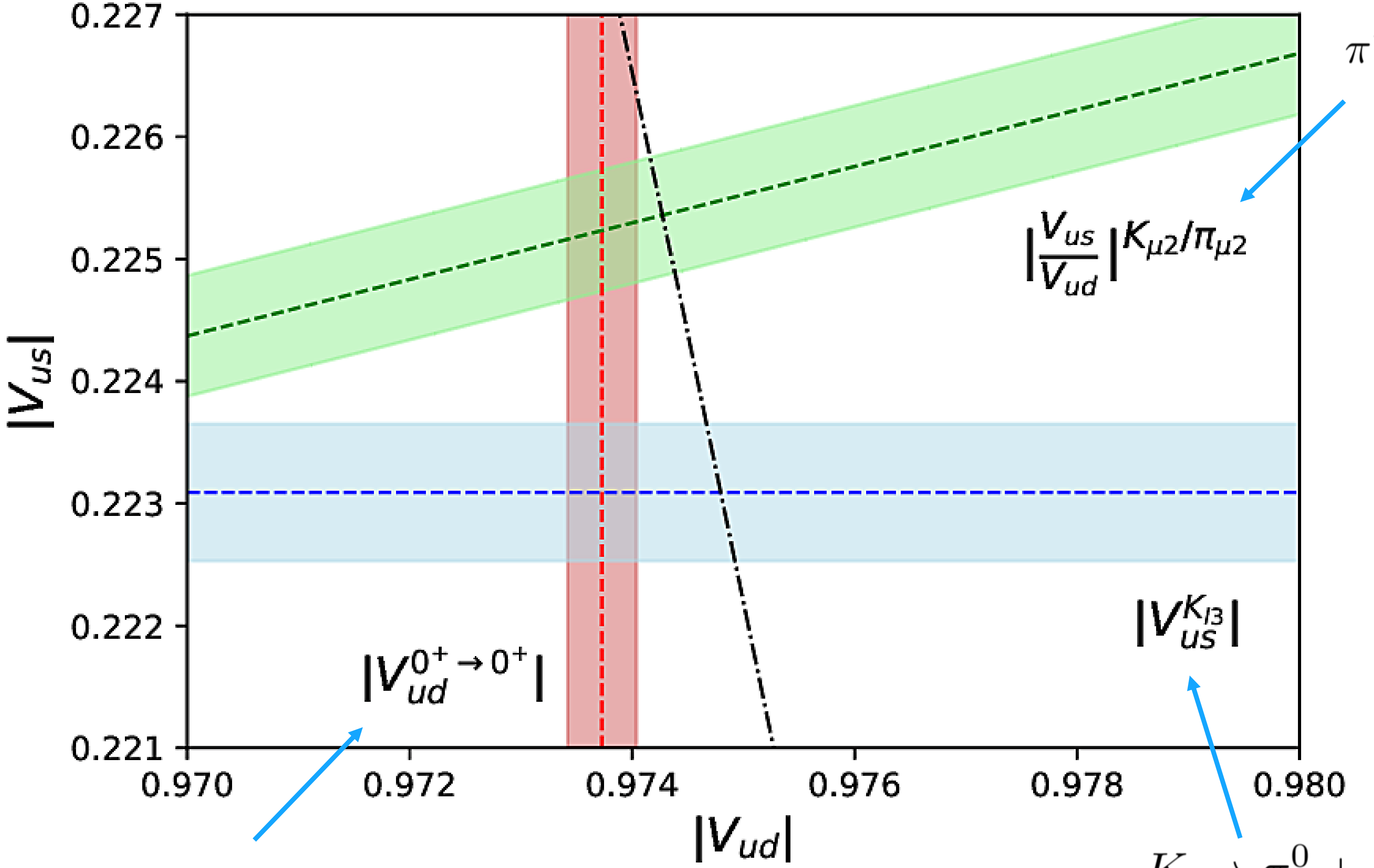
JOHANNES GUTENBERG UNIVERSITÄT MAINZ



Discovery, accelerated

$K \rightarrow \mu^+ + \nu_\mu$

$\pi^+ \rightarrow \mu^+ + \nu_\mu$



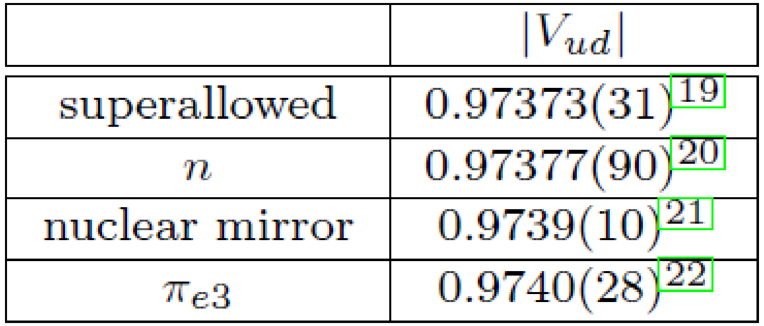
$|0^+\rangle \rightarrow |0^+\rangle$

$K \rightarrow \pi^0 + e^+ + \nu_e$

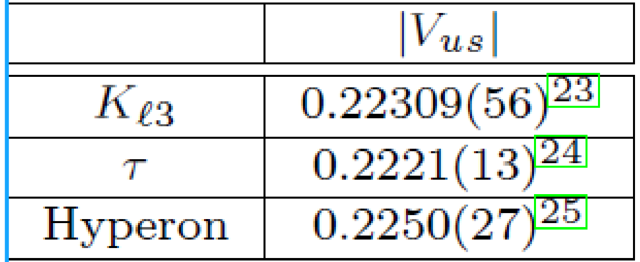
$K \rightarrow \pi^0 + \mu^+ + \nu_\mu$

Seng (2022) and references therein.

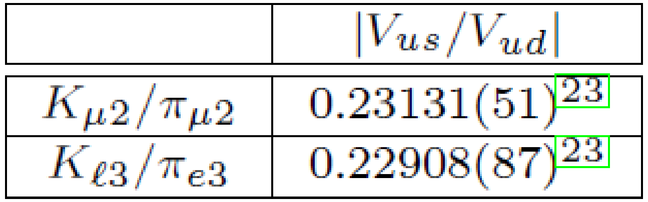
V_{ud} element of CKM matrix



	$\|V_{ud}\|$
superallowed	$0.97373(31)$[19]
n	$0.97377(90)$[20]
nuclear mirror	$0.9739(10)$[21]
π_{e3}	$0.9740(28)$[22]



	$\|V_{us}\|$
$K_{\ell 3}$	$0.22309(56)$[23]
τ	$0.2221(13)$[24]
Hyperon	$0.2250(27)$[25]

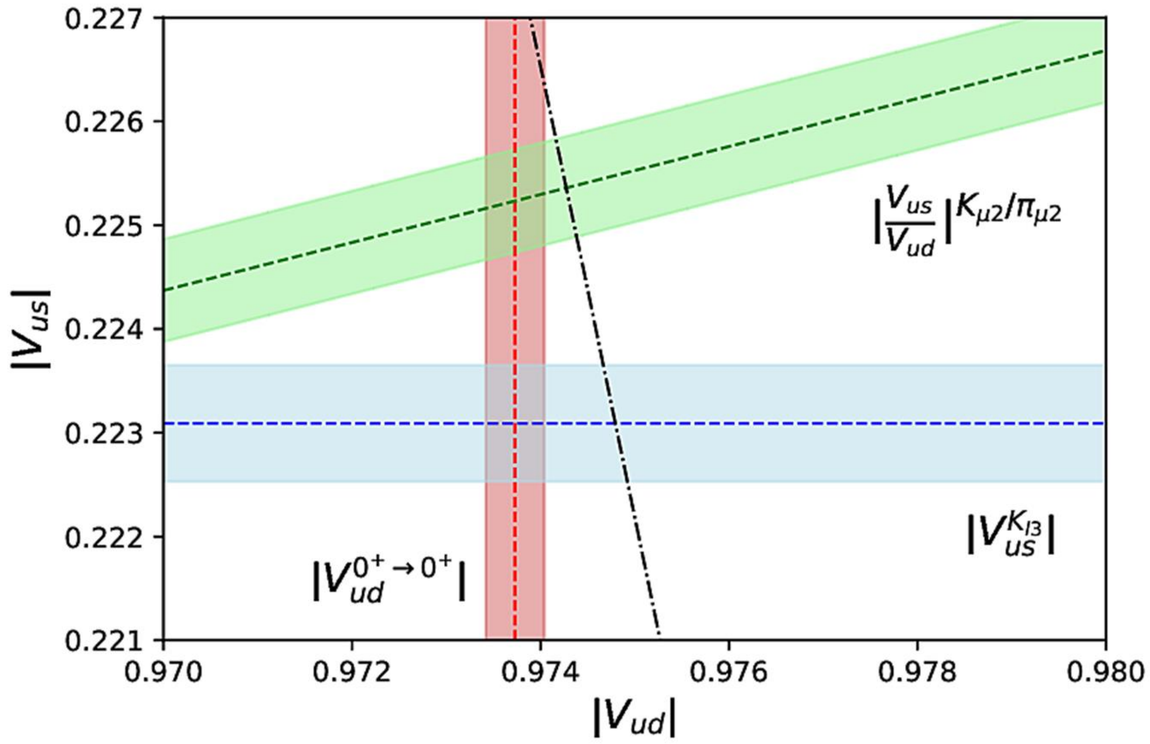


	$\|V_{us}/V_{ud}\|$
$K_{\mu 2}/\pi_{\mu 2}$	$0.23131(51)$[23]
$K_{\ell 3}/\pi_{e3}$	$0.22908(87)$[23]

$$|0^+\rangle \to |0^+\rangle$$

$$n \to p e^- \bar{\nu}_e$$

$$\pi^+ \to \pi^0 e^+ \nu_e(\gamma)$$



Beta decay in the Standard Model

$$\mathcal{L}_{\mathrm{CC}} = -\frac{G_F}{\sqrt{2}} \left(\bar{u}_L \quad \bar{c}_L \quad \bar{t}_L\right) \gamma^{\mu} W_{\mu}^{+} V_{\mathrm{CKM}} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + \mathrm{h.c.}$$

$$|V_{ud}|^2 = \frac{\hbar^7}{G_F^2 m_e^5 c^4} \frac{\pi^3 \ln(2)}{\mathcal{F}t(1+\Delta_R^V)}$$

$$|0^+\rangle \rightarrow |0^+\rangle$$

G_F ≡ Fermi coupling constant
determined from muon β decay

[2] Zyla et al. (2020)

$\mathrm{V_{ud}}$ element of CKM matrix

- Precise $\mathrm{V_{ud}}$ from superallowed Fermi transitions

$$|V_{ud}|^2 = \frac{\hbar^7}{G_F^2 m_e^5 c^4} \frac{\pi^3 \ln(2)}{\mathcal{F}t(1+\Delta_R^V)}$$

$G_F \equiv$ Fermi coupling constant
determined from muon β decay

  - hadronic matrix elements modified by nuclear environment
  - Fermi matrix element renormalized by isospin non–conserving forces

$$\mathcal{F}t = ft(1+\delta_R')\underline{(1-\delta_C+\delta_{NS})}$$

$$\mathcal{F}t = \frac{K}{G_V^2 |M_{F0}|^2 (1+\Delta_R^V)}$$

[2] Zyla et al. (2020)

Historical treatment

Pre-2018 (for almost 30 years)

- δ_{NS} from shell model and approximate single-nucleon currents
- δ_{C} from shell model with Woods-Saxon potential

$$T^{A}_{\mu\nu} \to \sum_k \langle f | \widetilde{J^{W}_{\mu}(k) G_{\mathrm{nuc}} J^{\mathrm{EM}}_{\nu}(k)} | i \rangle$$

$$\to \sum_k \langle f | \widetilde{J^{W}_{\mu}(k) [S_F \otimes G^{A''}_{\mathrm{nuc}}] J^{\mathrm{EM}}_{\nu}(k)} | i \rangle$$

$$\delta_{\mathrm{NS},A} = \frac{\alpha}{\pi}[q_A q_S^{(0)} - 1] C_{\mathrm{Born}}^{\mathrm{free}}$$

[3] Seng et al. (2018)
[4] Gorchtein et al. (2019)
[5] Hardy et al. (2020)

Historical treatment

Pre-2018 (for almost 30 years)

- δ_{NS} from shell model and approximate single-nucleon currents
- δ_{C} from shell model with Woods-Saxon potential

Since 2018

- Data-driven dispersion integral approach for Δ_R^V **[3-4]** which reduced radiative correction uncertainty by factor of ~ 2
- Formal theory for extraction of δ_{NS}
- Ongoing nuclear theory [this work] and lattice QCD calculations of electroweak box diagrams

[3] Seng et al. (2018)
[4] Gorchtein et al. (2019)
[5] Hardy et al. (2020)

Historical treatment

Pre-2

- δ_{NS}
- δ_{C} f

Since

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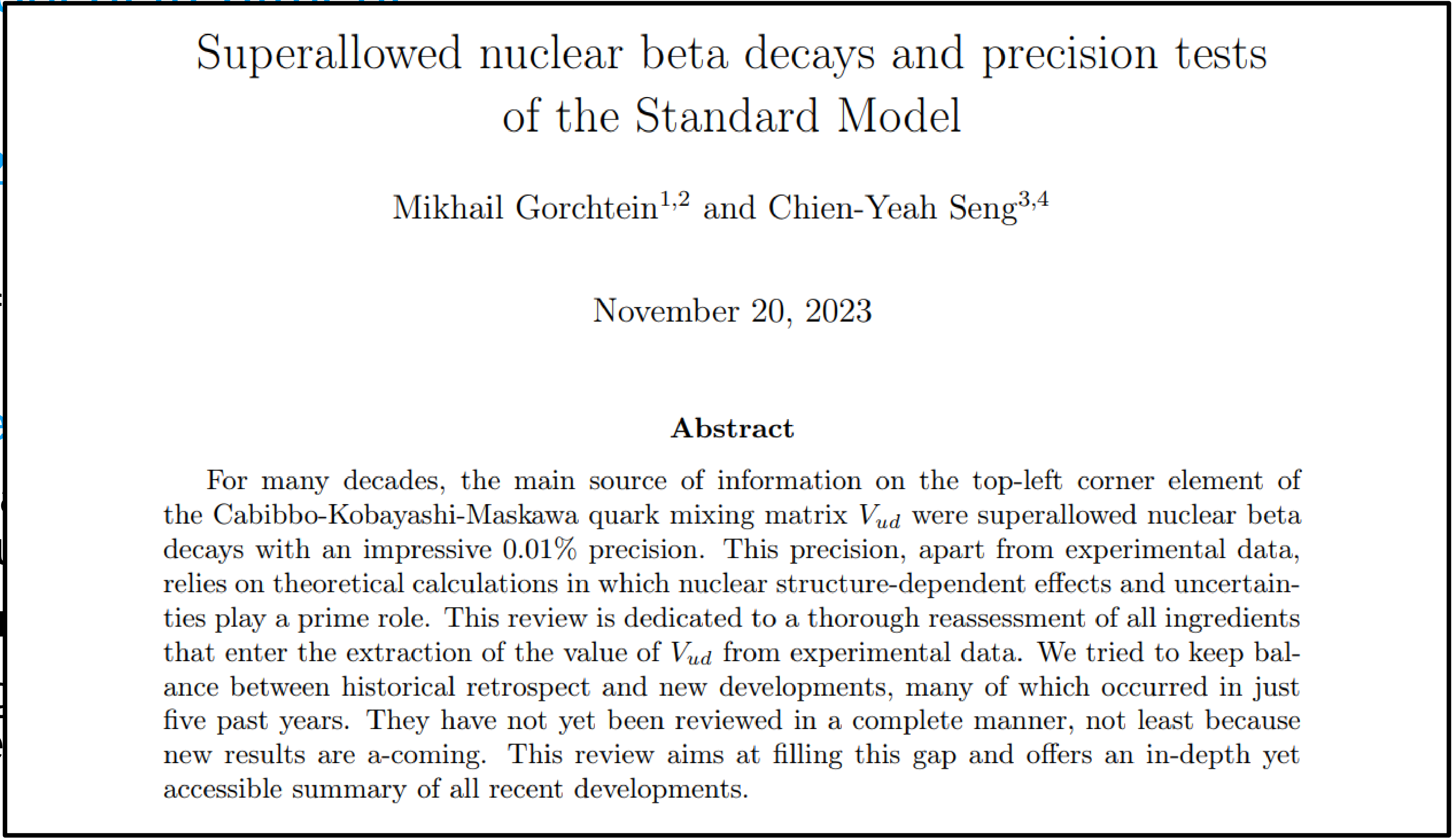


Superallowed nuclear beta decays and precision tests of the Standard Model

Mikhail Gorchtein[1,2] and Chien-Yeah Seng[3,4]

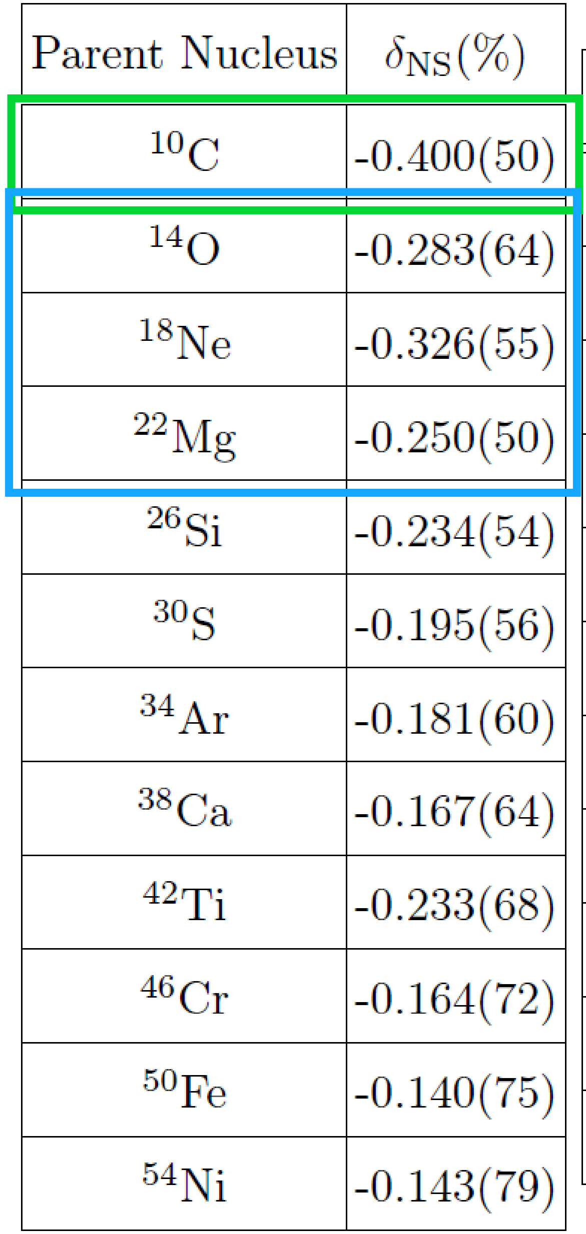
November 20, 2023

Abstract

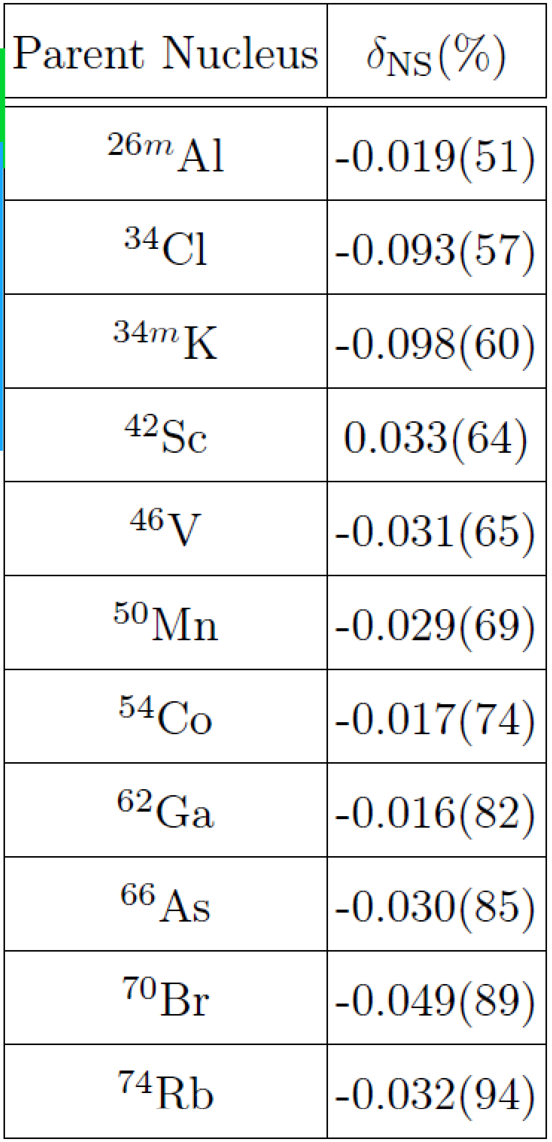
For many decades, the main source of information on the top-left corner element of the Cabibbo-Kobayashi-Maskawa quark mixing matrix V_{ud} were superallowed nuclear beta decays with an impressive 0.01% precision. This precision, apart from experimental data, relies on theoretical calculations in which nuclear structure-dependent effects and uncertainties play a prime role. This review is dedicated to a thorough reassessment of all ingredients that enter the extraction of the value of V_{ud} from experimental data. We tried to keep balance between historical retrospect and new developments, many of which occurred in just five past years. They have not yet been reviewed in a complete manner, not least because new results are a-coming. This review aims at filling this gap and offers an in-depth yet accessible summary of all recent developments.

[3] Seng et al. (2018)
[4] Gorchtein et al. (2019)
[5] Hardy et al. (2020)

NCSM



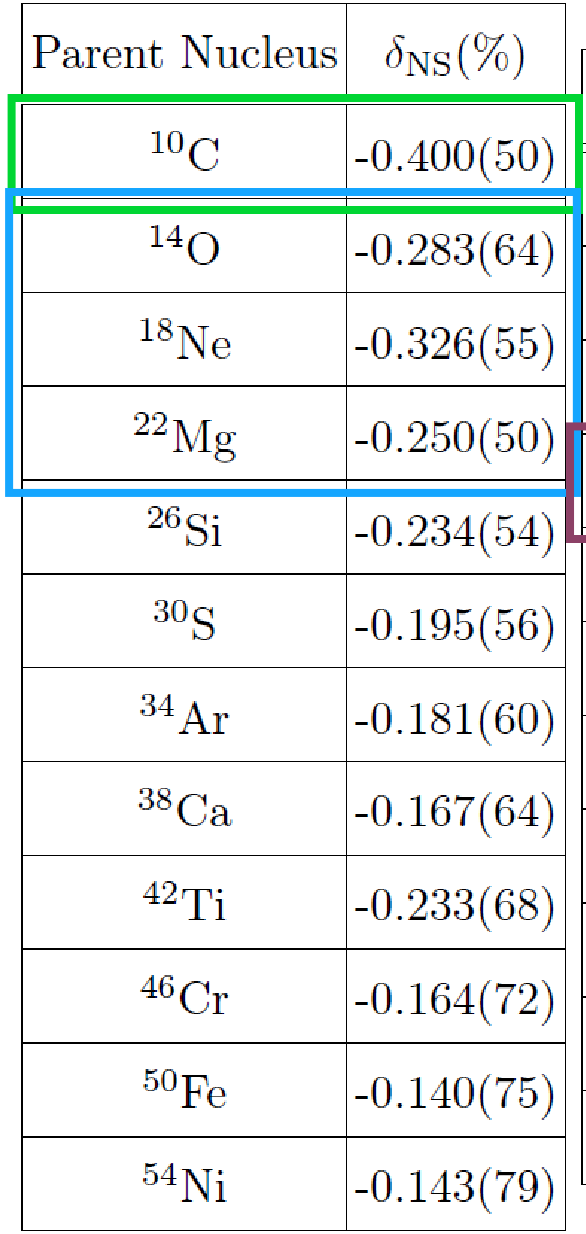
Parent Nucleus	$\delta_{\text{NS}}(\%)$
^{10}C	-0.400(50)
^{14}O	-0.283(64)
^{18}Ne	-0.326(55)
^{22}Mg	-0.250(50)
^{26}Si	-0.234(54)
^{30}S	-0.195(56)
^{34}Ar	-0.181(60)
^{38}Ca	-0.167(64)
^{42}Ti	-0.233(68)
^{46}Cr	-0.164(72)
^{50}Fe	-0.140(75)
^{54}Ni	-0.143(79)



Parent Nucleus	$\delta_{\text{NS}}(\%)$
^{26m}Al	-0.019(51)
^{34}Cl	-0.093(57)
^{34m}K	-0.098(60)
^{42}Sc	0.033(64)
^{46}V	-0.031(65)
^{50}Mn	-0.029(69)
^{54}Co	-0.017(74)
^{62}Ga	-0.016(82)
^{66}As	-0.030(85)
^{70}Br	-0.049(89)
^{74}Rb	-0.032(94)

[5] Hardy et al. (2020)

NCSM



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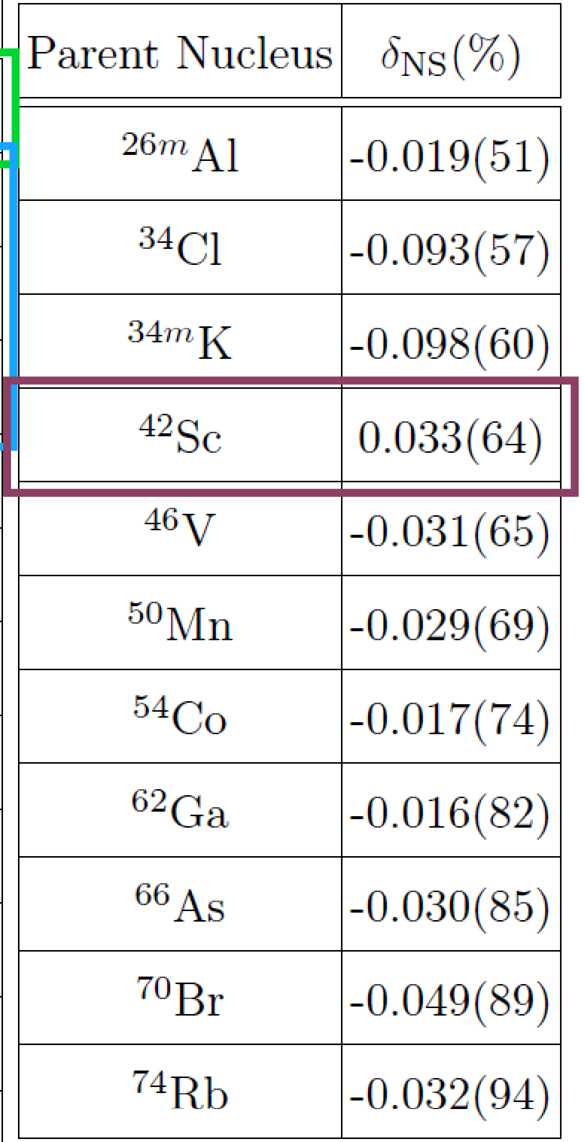

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CC

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NCSM

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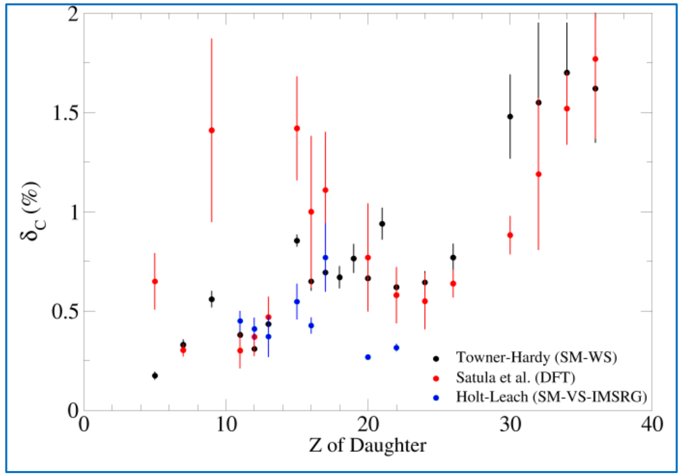
CC

VS–IMSRG



[5] Hardy et al. (2020)

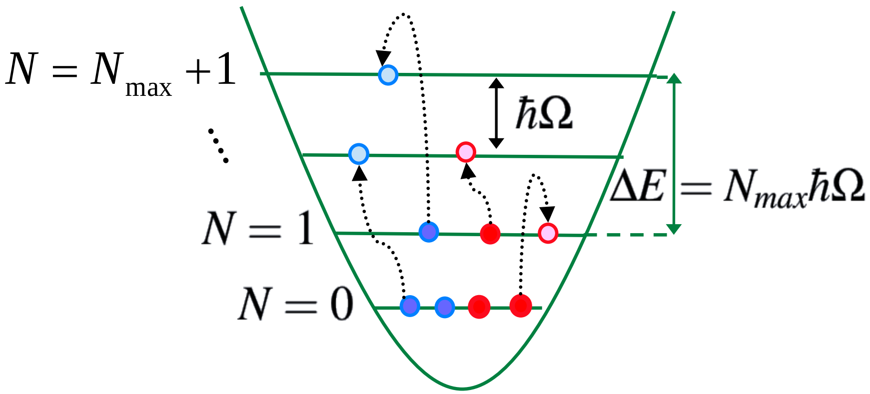
[6] Entem et al. (2017)
[7] Somà et al. (2020)

No-core shell model (NCSM)

- *Ab initio* approach to solving many-body Schrödinger equation
- Sole input are nuclear interactions from chiral effective field theory
  - NN-N^{4}LO(500) **[6]**
  - 3N(lnl)-N^{2}LO(650) **[7]**

$$H\left|\Psi_A^{J^\pi T}\right\rangle = E^{J^\pi T}\left|\Psi_A^{J^\pi T}\right\rangle$$

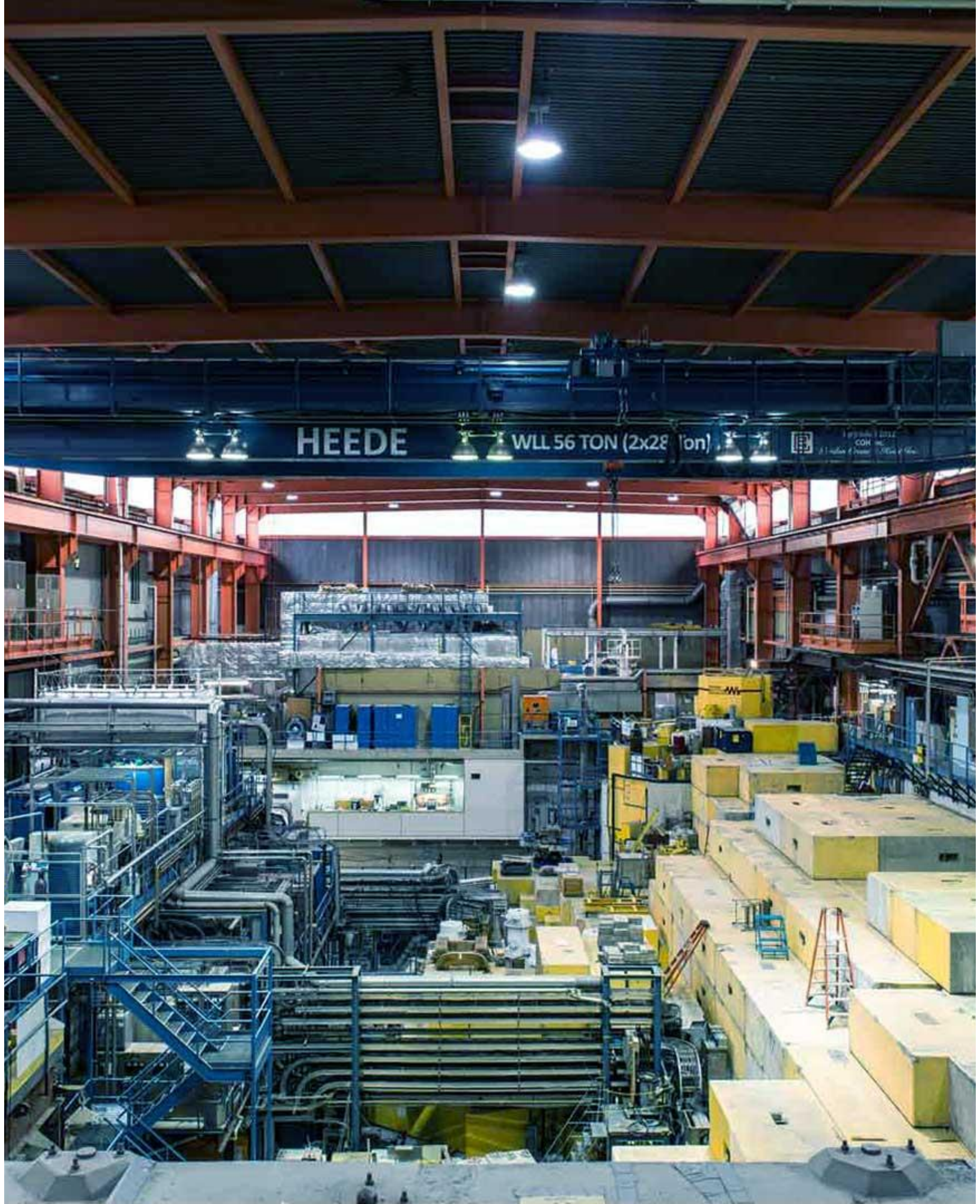
Anti-symmetrized products of many-body HO states



$$\left|\Psi_A^{J^\pi T}\right\rangle = \sum_{N=0}^{N_{max}} \sum_{\alpha} c_{N\alpha}^{J^\pi T}\left|\Phi_{N\alpha}^{J^\pi T}\right\rangle$$

$$\left|\Psi_A^{J^\pi T}\right\rangle_{\mathrm{SD}} = \sum_{N=0}^{N_{max}} \sum_{\alpha} [c^{(\mathrm{SD})}]_{N\alpha}^{J^\pi T}\left|\Phi_{N\alpha}^{J^\pi T}\right\rangle_{\mathrm{SD}} = \left|\Psi_A^{J^\pi T}\right\rangle \otimes \left|\Phi_{000}\right\rangle$$

Electroweak radiative correction δ_{NS}

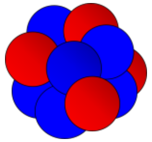


Discovery, accelerated

Standard Model

See Catharina's poster!

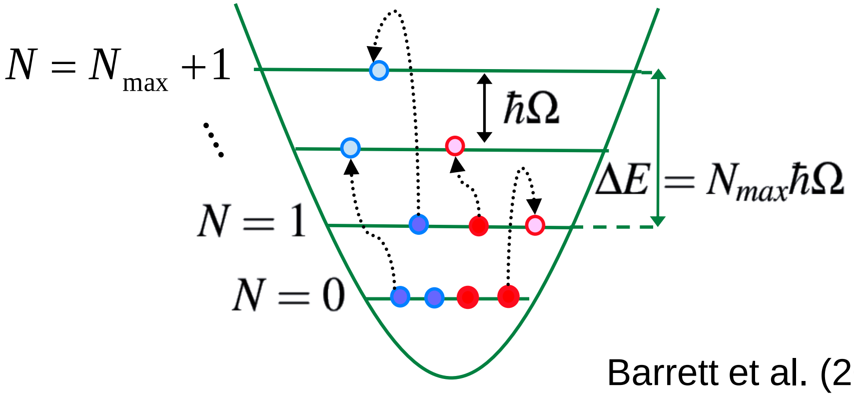
Chiral Effective Field Theory



Entem et al. (2017)
Somà et al. (2020)

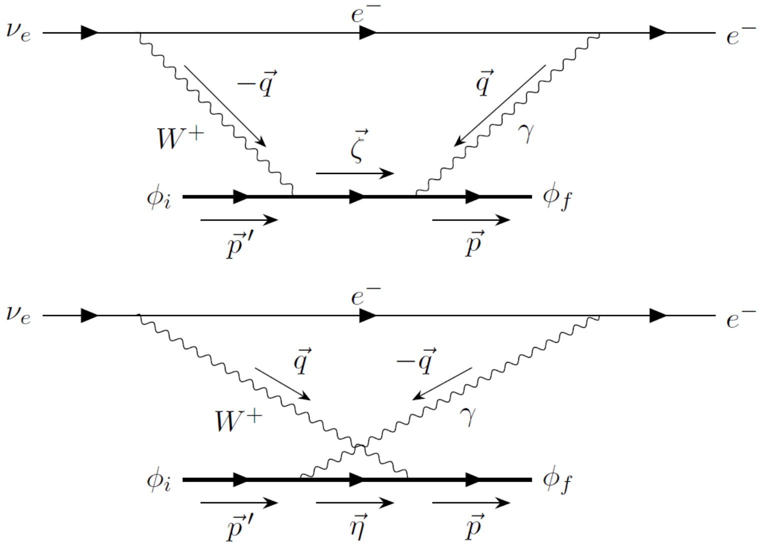
Weinberg (1991)
Epelbaum (2009)

$$H\left|\Psi_A^{J^\pi T}\right\rangle = E^{J^\pi T}\left|\Psi_A^{J^\pi T}\right\rangle$$



Barrett et al. (2013)
Haydock (1974)

E



- **Ultimate goal –** consistent chiral expansion for electroweak currents
- **For now –** leading multipole expansion

Haxton et al. (2007)
Seng et al. (2023)

Δ_R^V and δ_{NS}

Leptonic current

NME of charged weak current

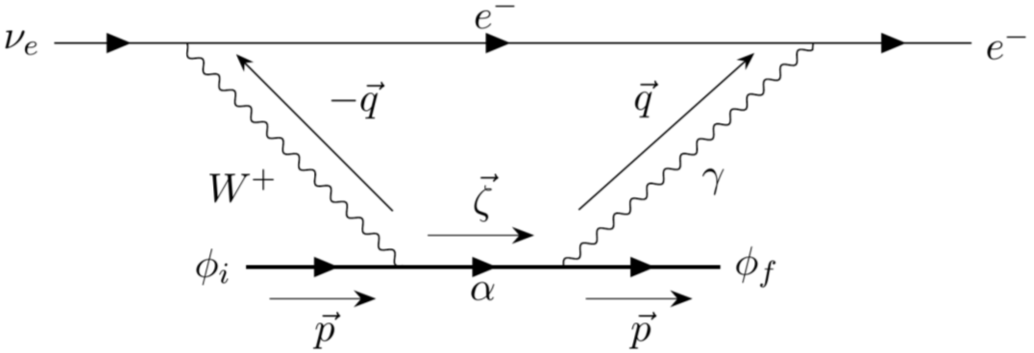
- Tree level beta decay amplitude

$$M_{tree} = -\frac{G_F}{\sqrt{2}} L_\lambda F^\lambda(p', p)$$

- Hadronic correction in forward scattering limit

$$\delta M = -i\sqrt{2} G_F e^2 L_\lambda \int \frac{d^4 q}{(2\pi)^4} \frac{M_W^2}{M_W^2 - q^2} \frac{\epsilon^{\mu\nu\alpha\lambda} q_\alpha}{[(p_e - q)^2 - m_e^2] q^2} \underline{T_{\mu\nu}(p', p, q)}$$

$$\boxed{\delta M = \Box_{\gamma W}(E_e) M_{tree}}$$



[8] Seng et al. (2023)

Δ_R^V and δ_{NS}

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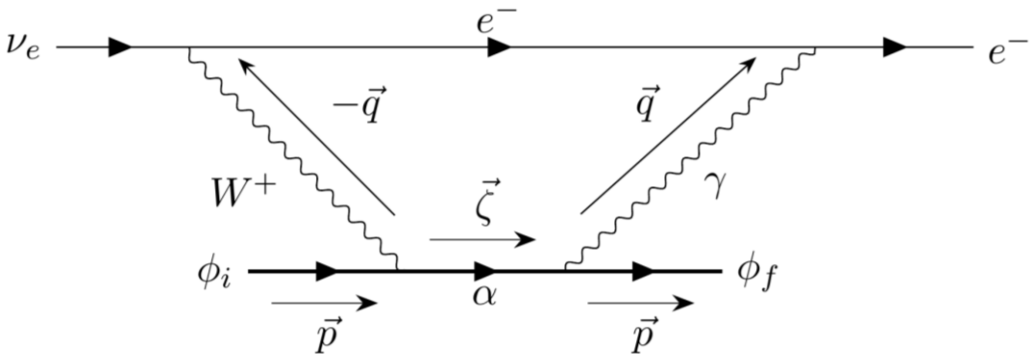


$$\boxed{\Box^b_{\gamma W}(E_e) = \frac{e^2}{M}\int \frac{d^4q}{(2\pi)^4}\frac{M_W^2}{M_W^2 - q^2}\frac{1}{q^2 + i\epsilon}\frac{1}{(p_e - q)^2 + i\epsilon'}\frac{M\nu\left(\frac{p_e\cdot q}{p\cdot p_e}\right) - q^2}{\nu}\frac{T_3(\nu, |\vec{q}|)}{f_+(0)}}$$

[8] Seng et al. (2023)

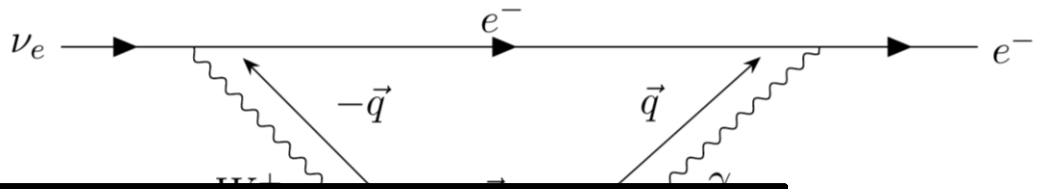
Δ_R^V and δ_{NS}

Leptonic current

NME of charged weak current

- Tree level beta decay amplitude $$M_{tree} = -\frac{G_F}{\sqrt{2}} L_\lambda F^\lambda(p', p)$$

- Hadronic correction in forward scattering limit



$$\delta M = \Box \, W(E \,) M$$

$$\delta_{NS} = 2[\Box^{\text{b,nuc}}_{\gamma W} - \Box^{\text{b,free n}}_{\gamma W}]$$

$$\Box^b_{\gamma W}(E_e) = \frac{e}{M} \int \frac{d\,q}{(2\pi)^4} \frac{M_W}{M_W^2 - q^2} \frac{1}{q^2 + i\epsilon} \frac{1}{(p_e - q)^2 + i\epsilon'} \frac{(p \cdot p_e) \quad 1}{\nu} \frac{T_3(\nu, |\vec{q}|)}{f_+(0)}$$

[8] Seng et al. (2023)

Lanczos subspace method

- Reformulate resolvent operator as inhomogeneous Schrödinger equation

$$(H - E\mathbb{1})\big|\Phi\big\rangle = \hat{O}\big|\Psi\big\rangle$$

$$\boxed{\begin{aligned}
H\mathbf{v}_1 &= \alpha_1\mathbf{v}_1 + \beta_1\mathbf{v}_2 \\
H\mathbf{v}_2 &= \beta_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \beta_2\mathbf{v}_3 \\
H\mathbf{v}_3 &= \qquad\qquad \beta_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \beta_3\mathbf{v}_4 \\
H\mathbf{v}_4 &= \qquad\qquad\qquad\qquad \beta_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \beta_4\mathbf{v}_5
\end{aligned}}$$

Select pivot as source term

$$\big|v_1\big\rangle = \frac{\hat{O}\big|\Psi\big\rangle}{\sqrt{\big\langle\Psi\big|\hat{O}^\dagger\hat{O}\big|\Psi\big\rangle}}$$

[9] Haydock (1974)
[10] Dagotto (1994)

Lanczos subspace method

- Reformulate resolvent operator as inhomogeneous Schrödinger equation

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H\mathbf{v}_4 &= \qquad\qquad\qquad\qquad \beta_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \beta_4\mathbf{v}_5
\end{aligned}}$$

- Resolvent amplitudes reconstructed via Lanczos basis
- Avoids (total) brute force calculation of intermediate states

[9] Haydock (1974)
[10] Dagotto (1994)

Lanczos subspace method

- Reformulate resolvent operator as inhomogeneous Schrödinger equation

$$(H - E\mathbb{1})\big|\Phi\big\rangle = \hat{O}\big|\Psi\big\rangle$$

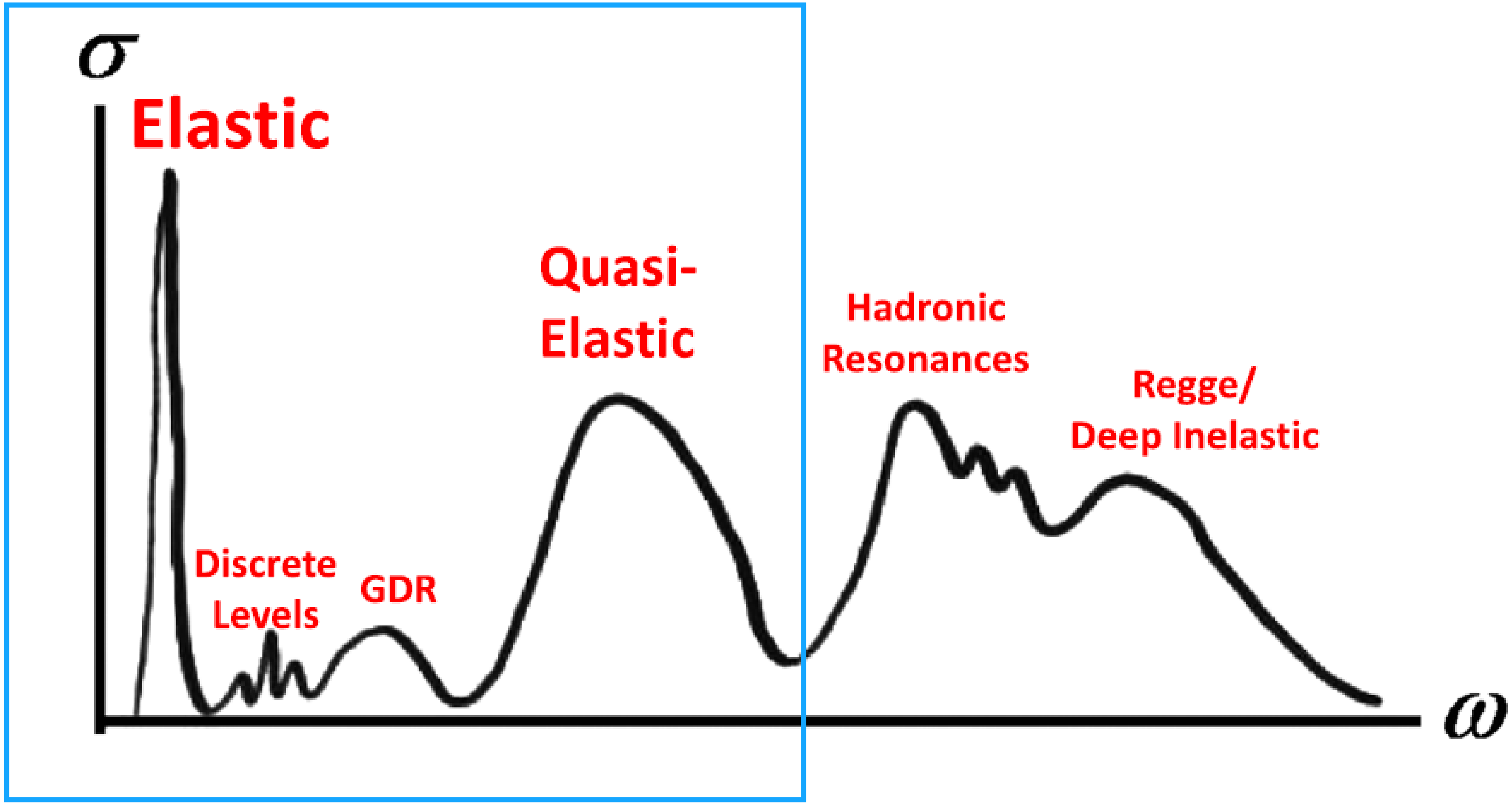
$$\boxed{\begin{aligned}
H\mathbf{v}_1 &= \alpha_1\mathbf{v}_1 + \beta_1\mathbf{v}_2 \\
H\mathbf{v}_2 &= \beta_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \beta_2\mathbf{v}_3 \\
H\mathbf{v}_3 &= \beta_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \beta_3\mathbf{v}_4 \\
H\mathbf{v}_4 &= \beta_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \beta_4\mathbf{v}_5
\end{aligned}}$$

Access dynamical properties!

$$\langle J_n|\mathbf{O_1}|J_i\rangle = (\mathbf{P}^{(i)})^{\dagger}_{0n}\,||\mathbf{O_1}\,|J_i\rangle||$$

$$\langle J_i|\mathbf{O_2}|J_n\rangle = ||\mathbf{O_2}\,|J_i\rangle||\sum\left(\mathbf{P}^{(i)}\right)_{mn}\left\langle\phi_0^{(f)}\middle|\phi_0^{(i)}\right\rangle$$

[9] Haydock (1974)
[10] Dagotto (1994)


σ
Elastic
Discrete
Levels
GDR
Quasi-
Elastic
Hadronic
Resonances
Regge/
Deep Inelastic
ω

Nuclear poles

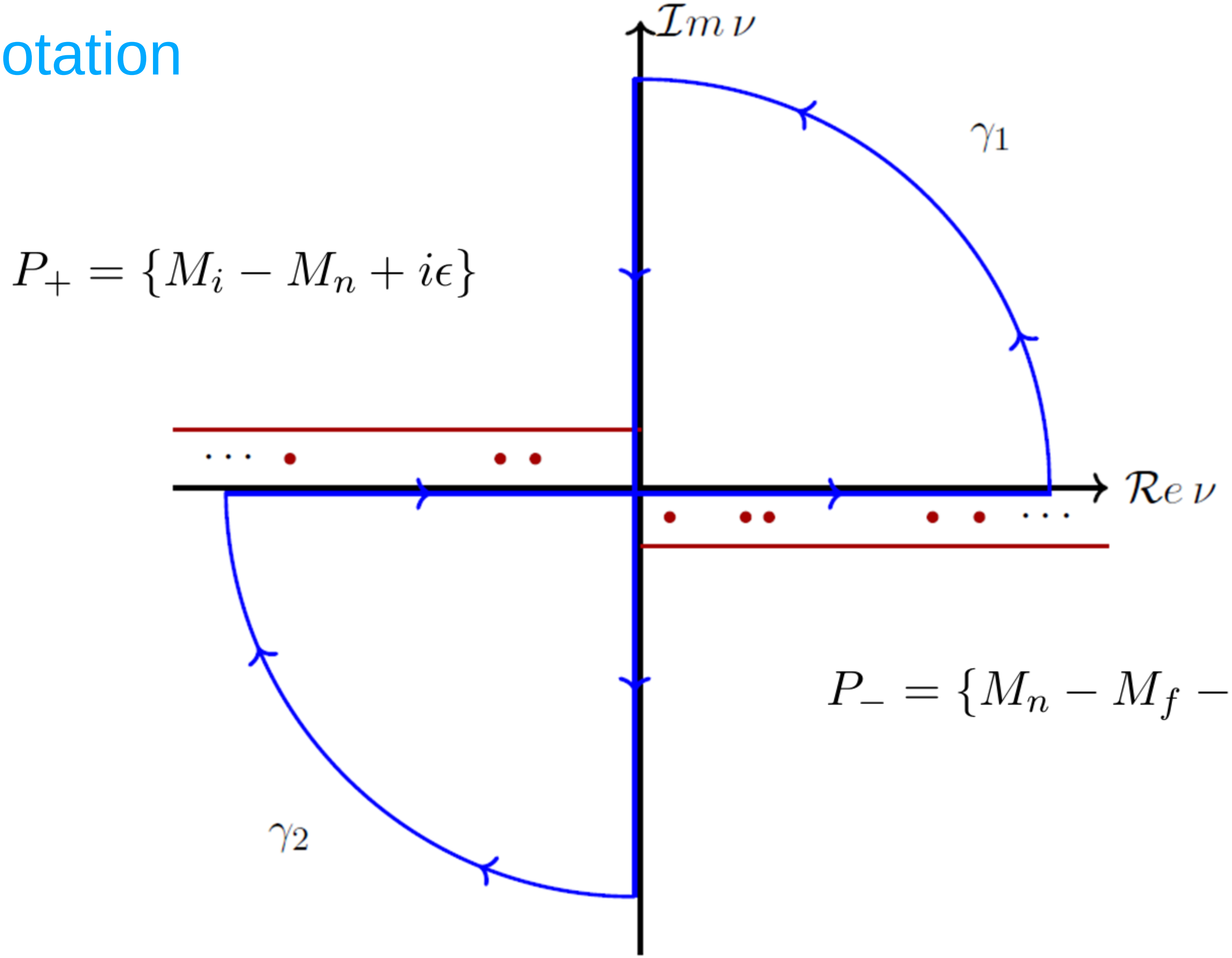
$$G(\nu + M_f + i\epsilon) = \sum_n \frac{|n\rangle\langle n|}{[\nu + M_f + i\epsilon] - M_n} \quad : \quad P_- = \{M_n - M_f - i\epsilon\}$$

$$G(-\nu + M_i + i\epsilon) = \sum_n \frac{|n\rangle\langle n|}{[-\nu + M_i + i\epsilon] - M_n} \quad : \quad P_+ = \{M_i - M_n + i\epsilon\}$$

- Numerical integration prone to instability
- Natural solution is Wick rotation

K.C. Greene

Wick rotation

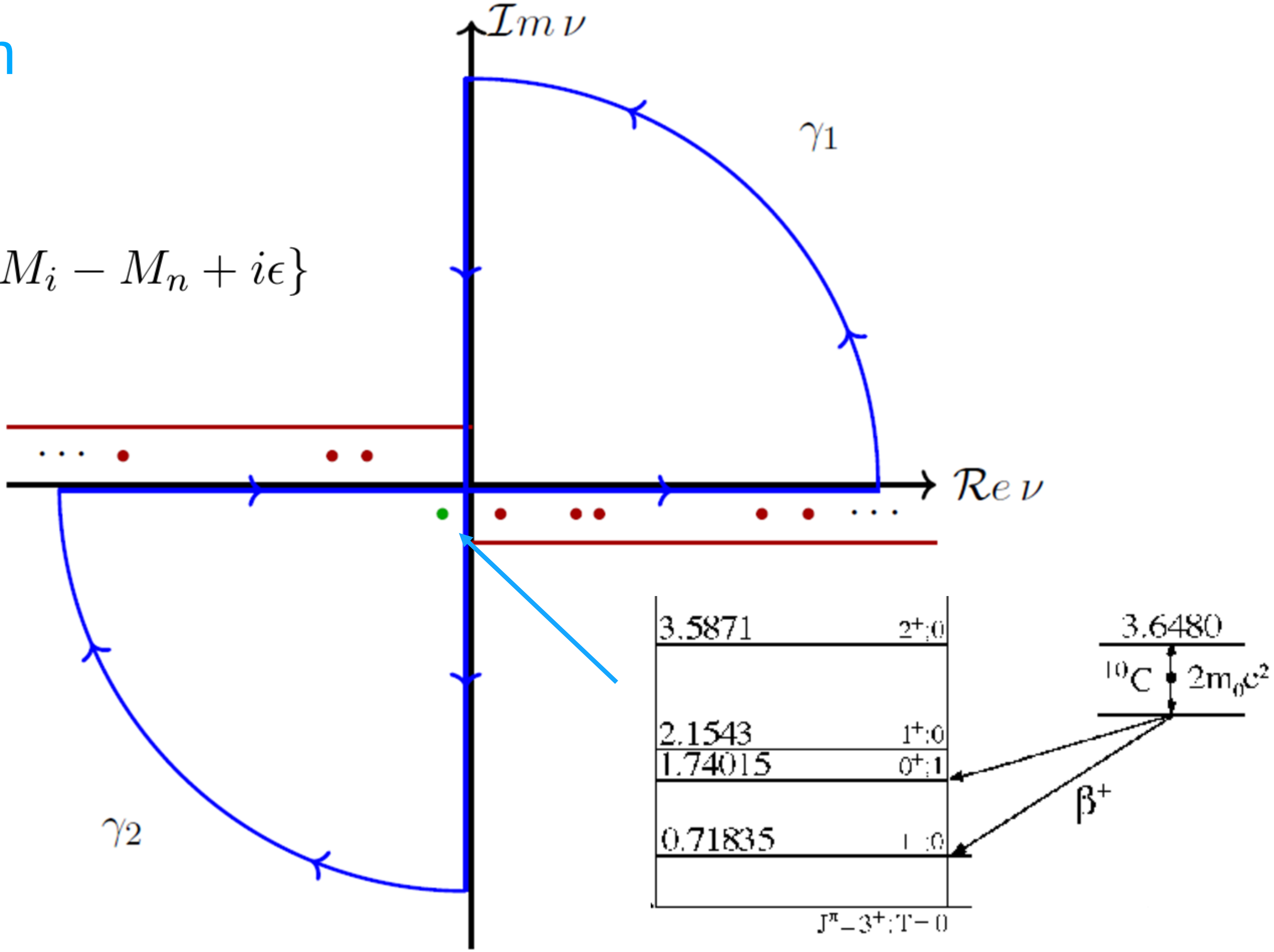


$P_+ = \{M_i - M_n + i\epsilon\}$

$P_- = \{M_n - M_f - i\epsilon\}$

Wick rotation

$$P_+ = \{M_i - M_n + i\epsilon\}$$



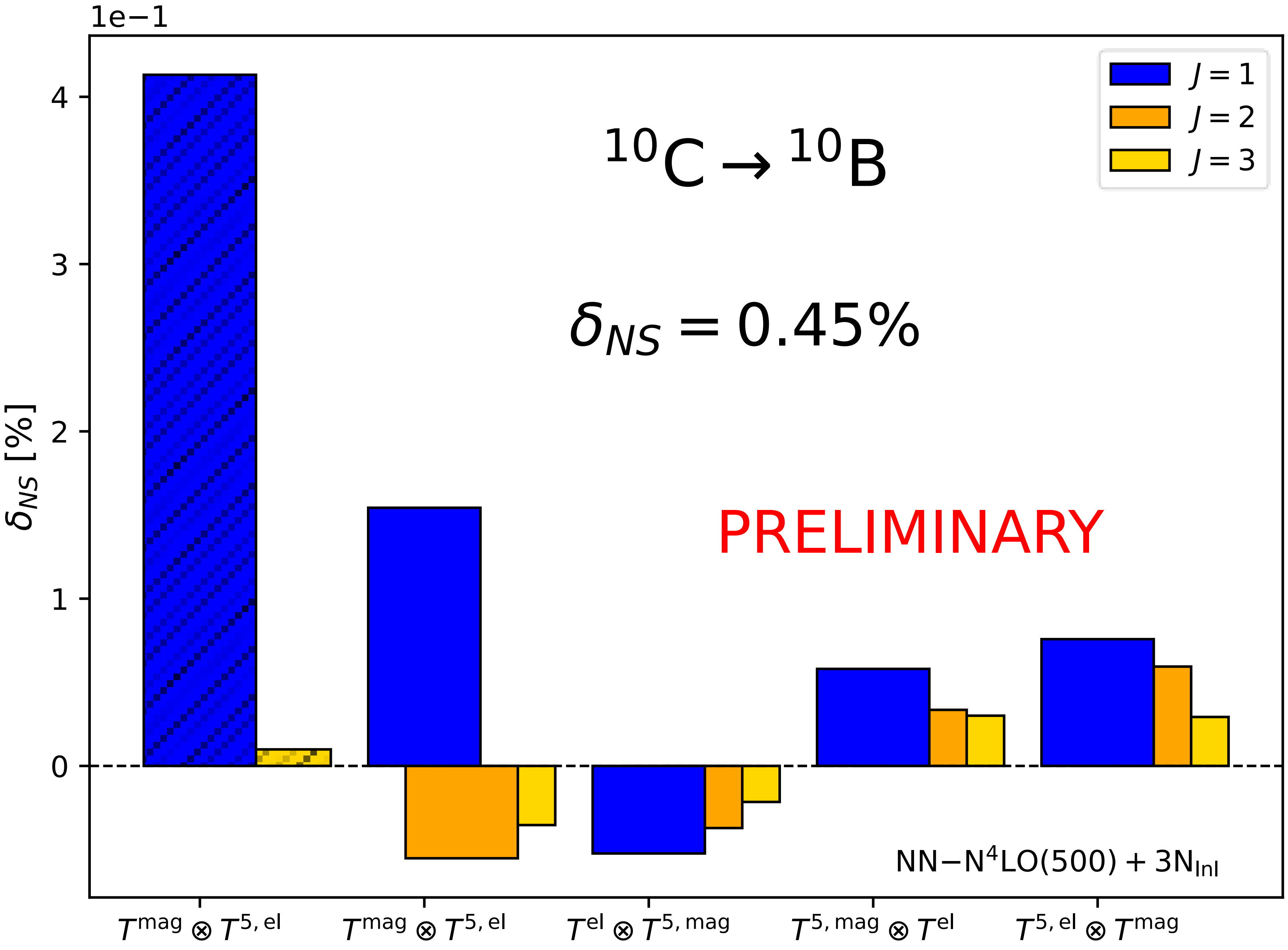
Electron energy expansion

$$\Box^b_{\gamma W}(E_e) = \underline{\left(\Box^b_{\gamma W}\right)_{\mathrm{Wick}}(E_e) + \left(\Box^b_{\gamma W}\right)_{\mathrm{Res},e}(E_e)} + \underline{\left(\Box^b_{\gamma W}\right)_{\mathrm{Res},T_3}(E_e)}$$

Wick rotated box diagram combined with electron propagator residue contribution regular at $E_e = 0$

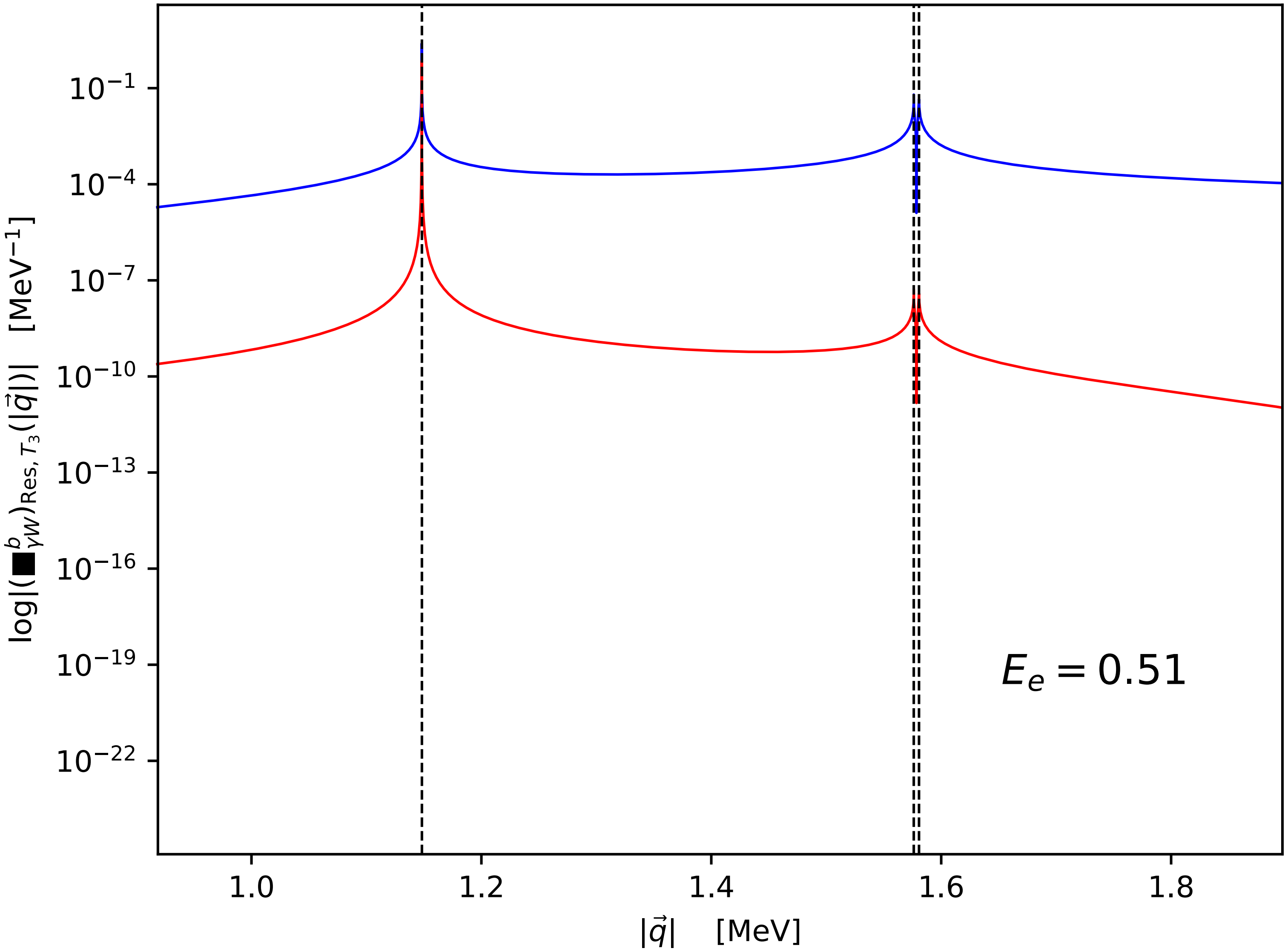
T_3 residue contribution **singular**

$$\boxed{\Box^b_{\gamma W}(E_e) = \boxminus_0 + E_e \boxminus_1 + \left(\Box^b_{\gamma W}\right)_{\mathrm{Res},T_3}(E_e) + \mathcal{O}(E_e^2)}$$


1e−1
δ_NS [%]
4
3
2
1
0
^{10}C → ^{10}B
δ_NS = 0.45%
PRELIMINARY
J = 1
J = 2
J = 3
NN−N^{4}LO(500) + 3N$_{lnl}$
$T^{mag} \otimes T^{5,el}$
$T^{mag} \otimes T^{5,el}$
$T^{el} \otimes T^{5,mag}$
$T^{5,mag} \otimes T^{el}$
$T^{5,el} \otimes T^{mag}$

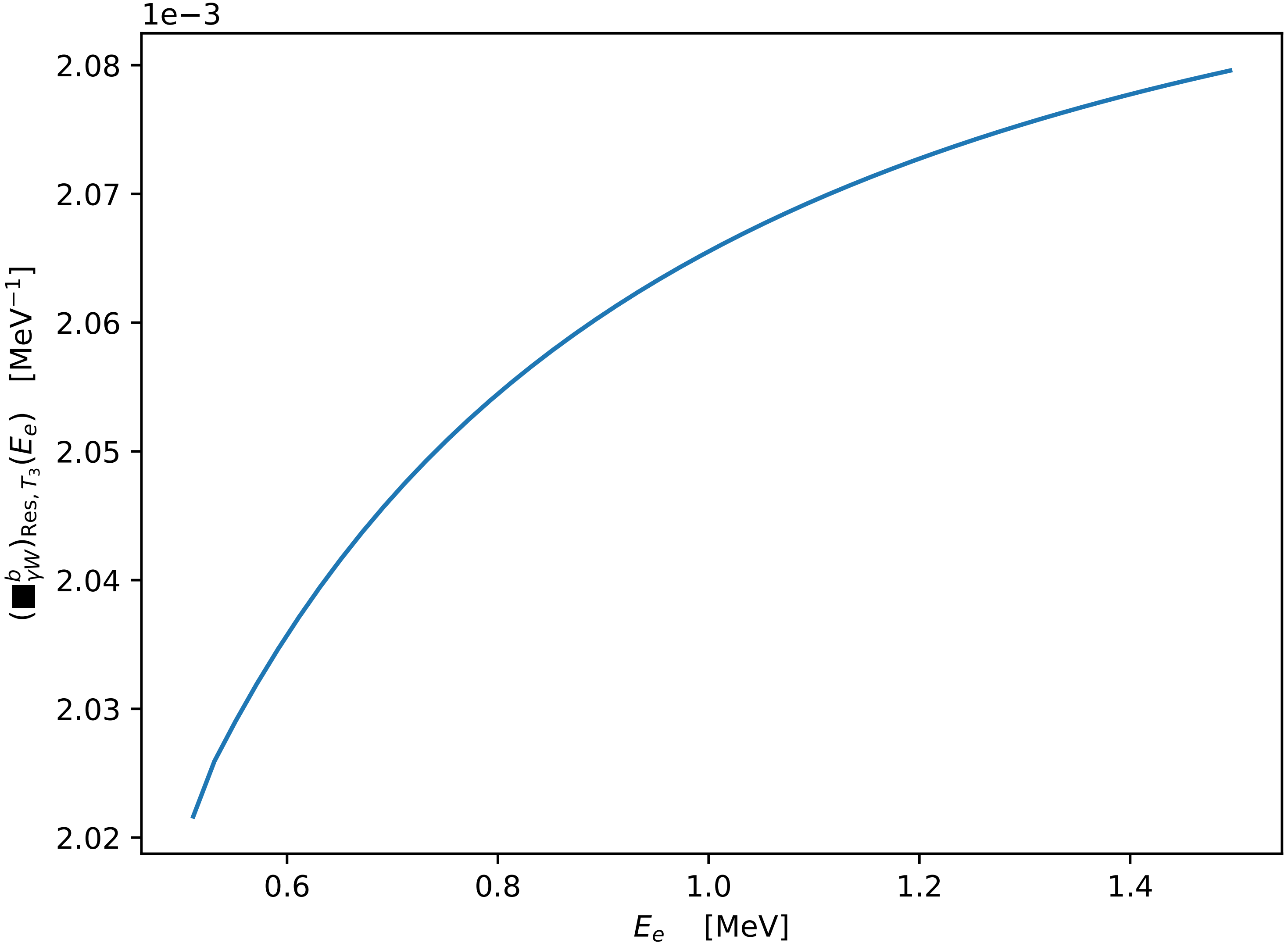
Compton residue

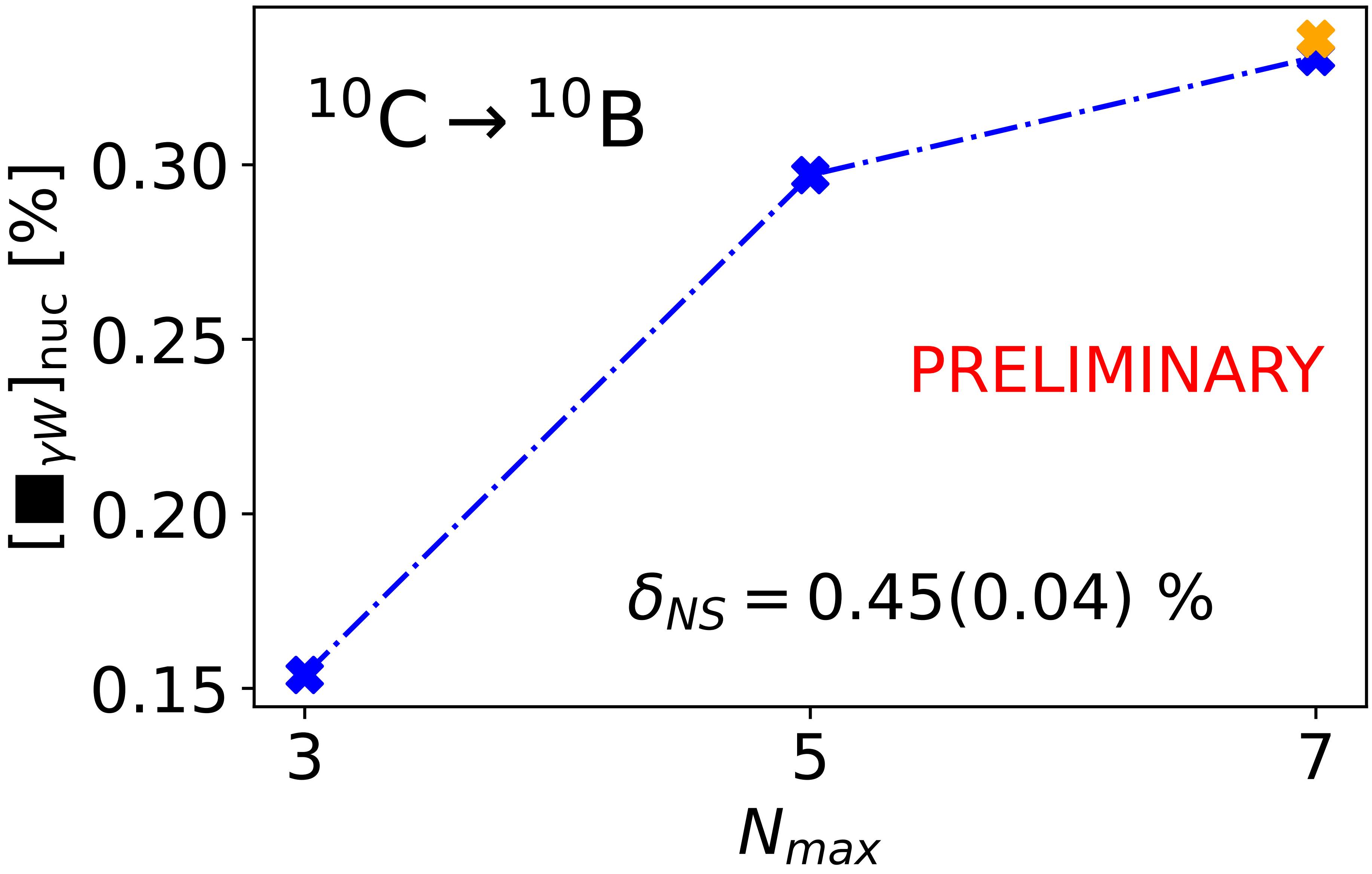
- Only needs transition amplitudes to low-lying eigenstates – simple!
- Residue integral contains additional poles in photon and electron propagators – careful numerical integration

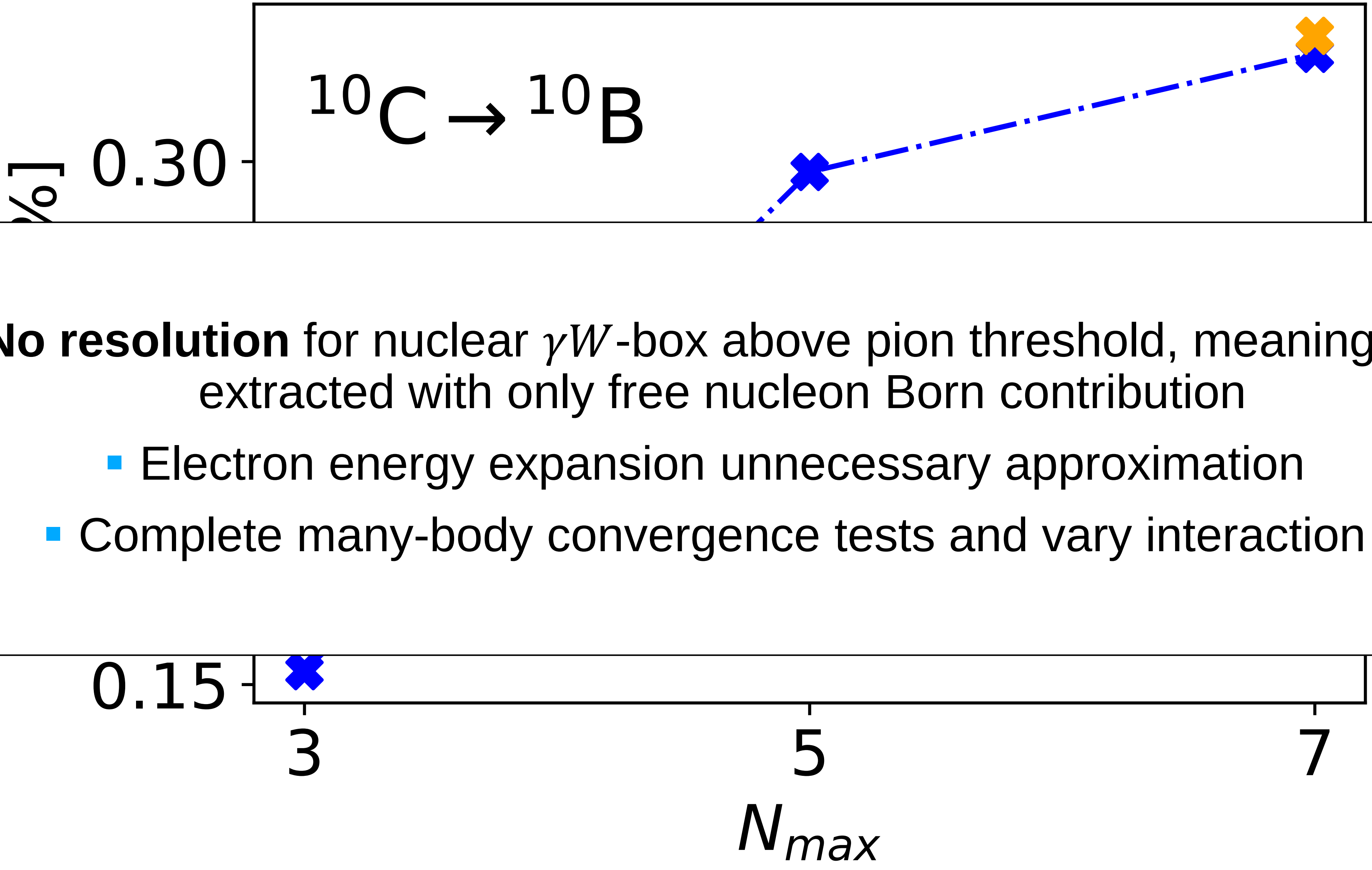


Compton residue

- Only needs transition amplitudes to low-lying eigenstates – simple!
- Residue integral contains additional poles in photon and electron propagators – careful numerical integration

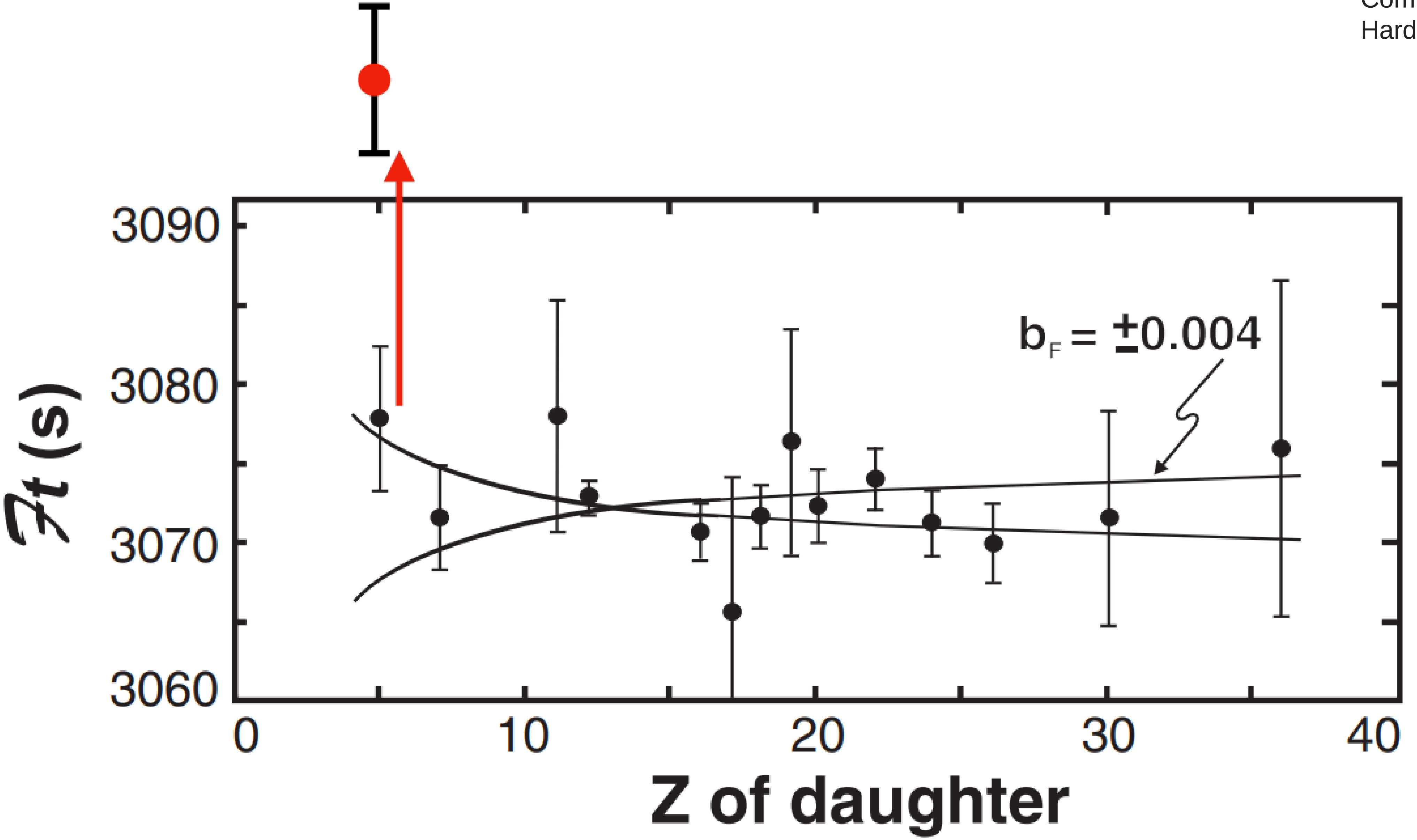



$^{10}C \to {}^{10}B$
PRELIMINARY
$\delta_{NS} = 0.45(0.04)$ %
$[■_{\gamma W}]_{nuc}$ [%]
0.30
0.25
0.20
0.15
3
5
7
N_{max}



- **No resolution** for nuclear γW-box above pion threshold, meaning δ_{NS} extracted with only free nucleon Born contribution
- Electron energy expansion unnecessary approximation
- Complete many-body convergence tests and vary interaction

M. Gorshteyn, Private Communication.
Hardy et al. (2020)



- **Goal:** consistent nuclear theory corrections to Fermi transitions
- Larger basis NCSM calculations of δ_{NS} complete
- First consistent NCSM calculation, seems that residue is dominant feature
- NCSMC calculations for δ_{C} ongoing with Mack Atkinson

Outlook

- Tackle large number of many-body calculations with realistic N_{max}
  - seperate inhomogeneous Schrödinger equation at each $|\vec{q}|$
  - $N_{|\vec{q}|} \times N_{terms} \times J_{max} = 50 \times 4 \times 3 = 600$ many body calculations
- Improve limited uncertainty quantification
- Heavier transitions, e.g., ^{14}O → ^{14}N

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Thank you
Merci

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Discovery, accelerated

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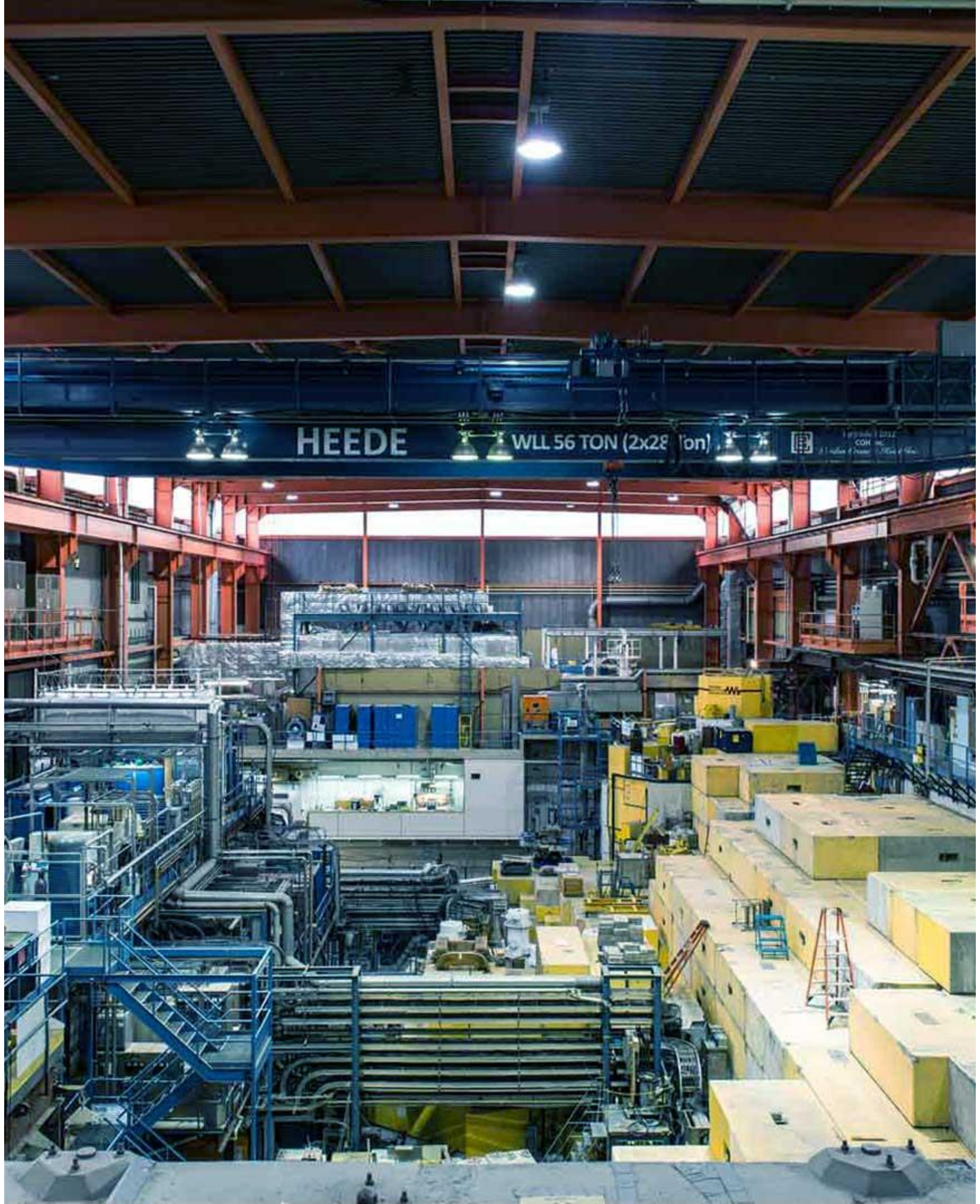
Backup slides for multipole expansion and δ_{NS}



Discovery, accelerated

Δ_R^V and δ_{NS}

Leptonic current

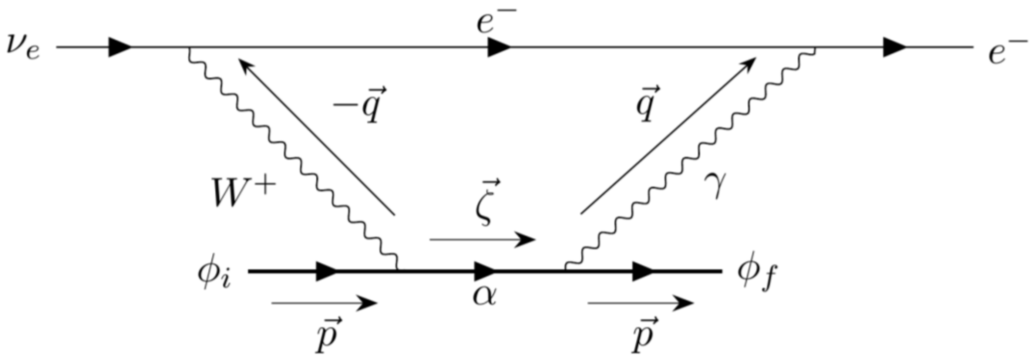
NME of charged weak current

- Tree level beta decay amplitude

$$M_{tree} = -\frac{G_F}{\sqrt{2}} L_\lambda F^\lambda(p', p)$$

- Hadronic correction in forward scattering limit

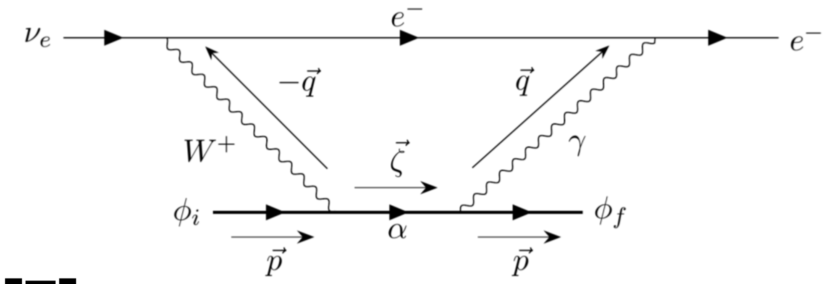
$$\delta_{NS} = 2[\Box_{\gamma W}^{\mathrm{b,nuc}} - \Box_{\gamma W}^{\mathrm{b,free\ n}}]$$



$$\boxed{T^{\mu\nu}(p,q) = \frac{1}{2}\int d^4x\ e^{iq\cdot x}\langle\phi_f(p)\left|T\left[J^\mu_{\mathrm{em}}(x)J^\nu_W(0)^\dagger\right]\right|\phi_i(p)\rangle}$$

[6] Seng et al. (2023)

Nonrelativistic Compton amplitude



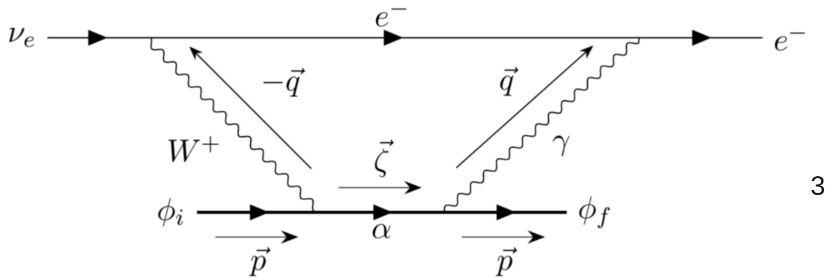
- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators

$$J^\mu(t,\vec{x}) = e^{-iHt} J^\mu(0,\vec{x})\, e^{iHt} \longrightarrow \boxed{G(E) = \sum_n \frac{|n\rangle\langle n|}{E - E_n}}$$

$$\begin{aligned} T^{\mu\nu}(p,q) = &-\frac{i}{2}\int d^3x\ e^{-i\vec{q}\cdot\vec{x}} \langle \phi_f(p) \big| J^\mu_{em}(0,\vec{x})\, \underline{G(M_f+\nu+i\epsilon)}\, J^{\dagger\,\nu}_W(0,\vec{0}) \big| \phi_i(p)\rangle \\ &-\frac{i}{2}\int d^3x\ e^{-i\vec{q}\cdot\vec{x}} \langle \phi_f(p) \big| J^{\dagger\,\nu}_W(0,\vec{0})\, \underline{G(M_i-\nu+i\epsilon)}\, J^\mu_{em}(0,\vec{x}) \big| \phi_i(p)\rangle \end{aligned}$$

[7] Haxton et al. (2007)

Nonrelativistic Compton amplitude



- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators
- Fourier transform currents into momentum space

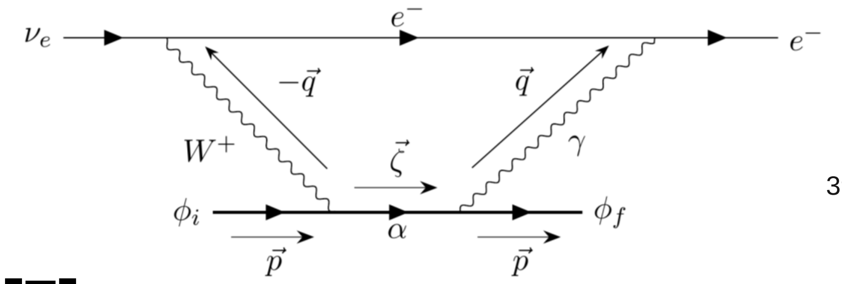
$$\boxed{J(\vec{r}) = \int \frac{d^3r}{(2\pi)^3}\; e^{i\vec{q}\cdot\vec{r}} J(\vec{q})} \quad +$$

Translation invariance

$$\begin{aligned} T^{\mu\nu}(p,q) = {} & i\sqrt{M_i M_f}\,\langle \Phi_f \big| \underline{J^{\mu}_{em}(\vec{q})}\, G(M_f + \nu + i\epsilon)\, \underline{J^{\dagger\,\nu}_{W}(-\vec{q})} \big| \Phi_i \rangle \\ & + i\sqrt{M_i M_f}\,\langle \Phi_f \big| \underline{J^{\dagger\,\nu}_{W}(-\vec{q})}\, G(M_i - \nu + i\epsilon)\, \underline{J^{\mu}_{em}(\vec{q})} \big| \Phi_i \rangle \end{aligned}$$

[7] Haxton et al. (2007)

Nonrelativistic Compton amplitude



- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators
- Fourier transform currents into momentum space
- General multipole expansion of currents

$$M_{JM}(q) := \int d^3r\ \mathcal{M}_{JM}(q,\vec{r})\,\rho(\vec{r})$$

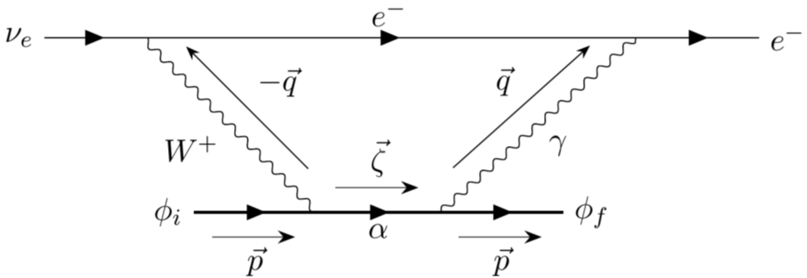
$$T^{\mathrm{el}}_{JM}(q) := \int d^3r\ \frac{1}{q}\left(\vec{\nabla}\times\vec{\mathcal{M}}^{M}_{JJ}(q,\vec{r})\right)\cdot\vec{J}(\vec{r})$$

$$L_{JM}(q) := \int d^3r\ \frac{i}{q}\left(\vec{\nabla}\mathcal{M}_{JM}(q,\vec{r})\right)\cdot\vec{J}(\vec{r})$$

$$T^{\mathrm{mag}}_{JM}(q) := \int d^3r\ \vec{\mathcal{M}}^{M}_{JJ}(q,\vec{r})\cdot\vec{J}(\vec{r})$$

[7] Haxton et al. (2007)

Nonrelativistic Compton amplitude

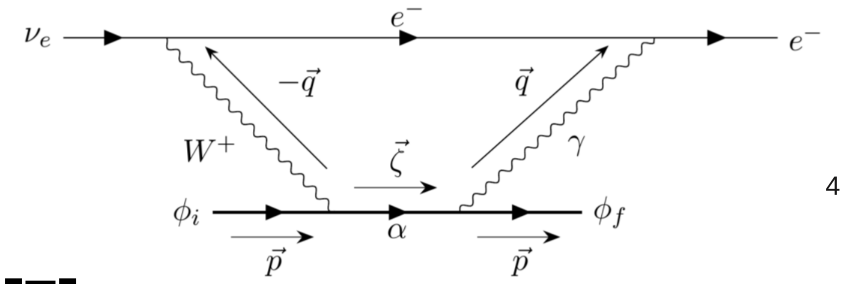


- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators
- Fourier transform currents into momentum space
- General multipole expansion of currents

$$T_3(\nu, |\vec{q}\,|) = 4\pi i \frac{\nu}{|\vec{q}\,|} \sqrt{M_i M_f} \sum_{J=1}^{\infty} (2J+1) \left\langle \Psi_f \right| \Big\{ T^{\mathrm{mag}}_{J0}\, G(\nu + M_f + i\epsilon)\, T^{5,\mathrm{el}}_{J0} + T^{\mathrm{el}}_{J0}\, G(\nu + M_f + i\epsilon)\, T^{5,\mathrm{mag}}_{J0}$$
$$+ T^{5,\mathrm{mag}}_{J0}\, G(-\nu + M_i + i\epsilon)\, T^{\mathrm{el}}_{J0} + T^{5,\mathrm{el}}_{J0}\, G(-\nu + M_i + i\epsilon) T^{\mathrm{mag}}_{J0} \Big\} (|\vec{q}\,|) \left| \Psi_i \right\rangle$$

[7] Haxton et al. (2007)

Nonrelativistic Compton amplitude



- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators
- Fourier transform currents into momentum space
- General multipole expansion of currents

Lanczos continued fraction method to compute nuclear Green's functions **[13-14]**

$$T_3(\nu,|\vec{q}\,|) = 4\pi i \frac{\nu}{|\vec{q}\,|}\sqrt{M_i M_f}\sum_{J=1}^{\infty}(2J+1)\,\langle\Psi_f|\Big\{T^{\text{mag}}_{J0}\boxed{G(\nu+M_f+i\epsilon)}T^{5,\text{el}}_{J0}+T^{\text{el}}_{J0}\boxed{G(\nu+M_f+i\epsilon)}T^{5,\text{mag}}_{J0}$$
$$+T^{5,\text{mag}}_{J0}\boxed{G(-\nu+M_i+i\epsilon)}T^{\text{el}}_{J0}+T^{5,\text{el}}_{J0}\boxed{G(-\nu+M_i+i\epsilon)}T^{\text{mag}}_{J0}\Big\}(|\vec{q}\,|)\,|\Psi_i\rangle$$

[7] Haxton et al. (2007)
[13] Hao et al. (2020)
[14] Froese et al. (2021)

Electron energy expansion

$$\boxed{\Box^{b}_{\gamma W}(E_e) = \boxminus_0 + E_e\,\boxminus_1 + \cdots + \left(\Box^{b}_{\gamma W}\right)_{\mathrm{Res},T_3}(E_e)}$$

$$\boxminus_0 = \frac{e^2}{M}\int\frac{d^3q}{(2\pi)^3}\int\frac{d\nu_E}{2\pi}\,\frac{M_W^2}{M_W^2-q^2}\,\frac{|\vec{q}|^2}{\nu_E(q^2+i\epsilon_1)^2}\,\frac{T_3\left(i\nu_E,|\vec{q}|\right)}{f_+(0)}$$

$$\boxminus_1 = \frac{8}{3}\,\frac{e^2}{M}\int\frac{d^3q}{(2\pi)^3}\int\frac{d\nu_E}{2\pi}\,\frac{M_W^2}{M_W^2-q^2}\,\frac{|\vec{q}|^2}{(q^2+i\epsilon_1)^3}\,\frac{iT_3\left(i\nu_E,|\vec{q}|\right)}{f_+(0)}$$

Multipole expansion of amplitude

$$J^\mu(q) = \Big(\rho(\vec{q}), \vec{J}(\vec{q})\Big) \qquad \longrightarrow \qquad \vec{J}(\vec{q}) = \sum_\lambda J(\vec{q},\lambda)\,\vec{\epsilon}^{\,*}_\lambda$$

$$e^{-i\vec{q}\cdot\vec{r}} = 4\pi \sum_{J=0}^{\infty}\sum_{M_J}(-i)^J j_J(qr)\,Y_{JM_J}(\hat{q})Y^*_{JM_J}(\hat{q})$$

$$\mathcal{M}_{JM}(q,\vec{r}) = j_J(qr)\,Y_{JM}(\hat{r}) \qquad\qquad \vec{\mathcal{M}}^M_{JL}(q,\vec{r}) = j_L(qr)\,\vec{Y}^M_{JL1}(\hat{r})$$

[7] Walecka (2004)

Multipole expansion of amplitude

$$\rho(\vec{q}) = \sqrt{4\pi} \sum_{J=0}^{\infty} (-i)^J \sqrt{2J+1}\, M_{J0}(q)$$

$$J(\vec{q}, \lambda = 0) = \sqrt{4\pi} \sum_{J=0}^{\infty} (-i)^J \sqrt{2J+1}\, L_{J0}(q)$$

$$J(\vec{q}, \lambda = \pm 1) = -\sqrt{2\pi} \sum_{J=1}^{\infty} (-i)^J \sqrt{2J+1} \left(\lambda T^{\mathrm{mag}}_{J\lambda}(q) - T^{\mathrm{el}}_{J\lambda}(q) \right)$$

Nuclear matrix elements of multipole operators

$$\left\langle N(p_f s_f m_{T_f})\middle| V^{\mu}_{TM_T}(0) \middle| N(p_i s_i m_{T_i})\right\rangle = \bar{u}_{s_f}(p_f)\left[\boxed{F_1^{(T)}}\gamma^{\mu} + \frac{\boxed{iF_2^{(T)}}}{2m_N}\sigma^{\mu\nu}(p_f - p_i)_{\nu}\right]u_{s_i}(p_i)\left\langle m_{T_f}\middle|\Gamma_{TM_T}\middle|m_{T_i}\right\rangle$$

$$\left\langle N(p_f s_f m_{T_f})\middle| A^{\mu}_{TM_T}(0) \middle| N(p_i s_i m_{T_i})\right\rangle = \bar{u}_{s_f}(p_f)\left[\boxed{G_A^{(T)}}\gamma^{\mu}\gamma_5 - \frac{\boxed{G_P^{(T)}}}{2m_N}\gamma_5(p_f - p_i)^{\mu}\right]u_{s_i}(p_i)\left\langle m_{T_f}\middle|\Gamma_{TM_T}\middle|m_{T_i}\right\rangle$$

$$\mathcal{M}_{JM}(q,\vec{r}) = j_J(qr)\,Y_{JM}(\hat{r})$$

$$\Delta_{JM}(q,\vec{r}) := \vec{\mathcal{M}}^{M}_{JJ}(q,\vec{r}) \cdot \frac{1}{q}\vec{\nabla}$$

$$\Delta'_{JM}(q,\vec{r}) := -i\left(\frac{1}{q}\vec{\nabla} \times \vec{\mathcal{M}}^{M}_{JJ}(q,\vec{r})\right) \cdot \frac{1}{q}\vec{\nabla}$$

$$\Sigma_{JM}(q,\vec{r}) := \vec{\mathcal{M}}^{M}_{JJ}(q,\vec{r}) \cdot \vec{\sigma}$$

$$\Sigma'_{JM}(q,\vec{r}) := -i\left(\frac{1}{q}\vec{\nabla} \times \vec{\mathcal{M}}^{M}_{JJ}(q,\vec{r})\right) \cdot \vec{\sigma}$$

$$\Sigma''_{JM}(q,\vec{r}) := \left(\frac{1}{q}\vec{\nabla}\mathcal{M}_{JM}(q,\vec{r})\right) \cdot \vec{\sigma}$$

$$\Omega_{JM}(q,\vec{r}) := \left(\mathcal{M}_{JM}(q,\vec{r})\ \vec{\sigma}\right) \cdot \frac{1}{q}\vec{\nabla}$$

Symmetry tests of T_3 amplitude

- Time reversal symmetry with exact isospin gives NME constraint
- Previously **assumed** nuclear T_3 matched nucleonic system

Nuclei	Nucleons	Pions
$\boxed{\begin{gathered} T_3^{(0)}(-\nu,Q^2) = -T_3^{(0)}(\nu,Q^2) \\ T_3^{(1)}(-\nu,Q^2) = \cdots \end{gathered}}$	$T_3^{(0)}(-\nu,Q^2) = -T_3^{(0)}(\nu,Q^2)$ $T_3^{(1)}(-\nu,Q^2) = T_3^{(1)}(\nu,Q^2)$	$T_3^{(0)}(-\nu,Q^2) = -T_3^{(0)}(\nu,Q^2)$ $T_3^{(1)}(\nu,Q^2) = 0$

$$\left\langle {}^{10}\mathrm{B}\middle| T_{J0}^{\mathrm{mag},(0)}(q)\, G(M+i\epsilon)\, T_{J0}^{5,\mathrm{el}}(q) \middle| {}^{10}\mathrm{C}\right\rangle = \left\langle {}^{10}\mathrm{C}\middle| T_{J0}^{\mathrm{mag},(0)}(q)\, G(M+i\epsilon)\, \tilde{T}_{J0}^{5,\mathrm{el}}(q) \middle| {}^{10}\mathrm{B}\right\rangle$$

$$\left\langle {}^{10}\mathrm{B}\middle| T_{J0}^{\mathrm{el},(0)}(q)\, G(M+i\epsilon)\, T_{J0}^{5,\mathrm{mag}}(q) \middle| {}^{10}\mathrm{C}\right\rangle = \left\langle {}^{10}\mathrm{C}\middle| T_{J0}^{\mathrm{el},(0)}(q)\, G(M+i\epsilon)\, \tilde{T}_{J0}^{5,\mathrm{mag}}(q) \middle| {}^{10}\mathrm{B}\right\rangle$$

[6] Seng et al. (2023)