The polarization propagator

in the self-consistent Gorkov-Green's function method:

Perturbation Theory

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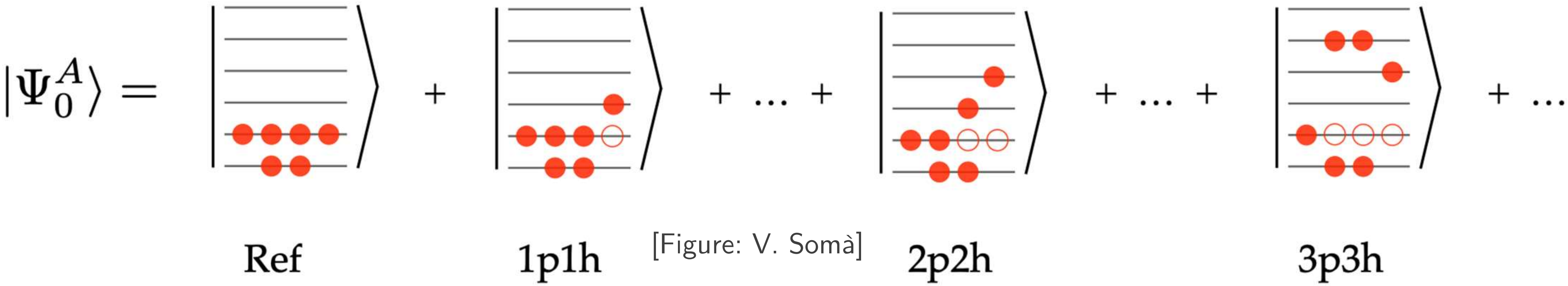
Self-Consistent Gorkov-Green's Function Theory

- Ab-initio approach, extending the **S**elf-**c**onsistent **G**reen's **f**unction theory to semimagic nuclei. In SCGF, the Z ***protons*** and N ***neutrons*** interact through realistic nuclear potentials, drawn from ***Chiral Effective Field Theory*** (χEFT)

Practically, χ^{EFT} forces are preprocessed via the ***similarity renormalization group****, in order to quench the coupling between low and high momenta in the Hamiltonian*

SCGGF adopts an efficient approximation scheme for the nuclear wavefunction, entailing ***a polynomial scaling*** in the size M of the space of single-particle excitations M^{α} with $\alpha \geq 4$

- ***Correlation-expansion methods:*** expansion of the exact nuclear wavefunction into the space of particle-hole excitations built through the correlator Ω on a given *reference state*:



[Figure: V. Somà]

$$|\Psi_0^A\rangle = \Omega|\Phi_0^A\rangle = |\Phi_0^A\rangle + |\Phi_0^{A\,1p1h}\rangle + \ldots + |\Phi_0^{A\,2p2h}\rangle + \ldots + |\Phi_0^{A\,3p3h}\rangle + \ldots$$

and the reference state Φ_0^A is the ground state of H_0, a solvable Hamiltonian, splitting the original one into $H = H_0 + H_I$ where H_I contains the 2-, 3-, ... -body interactions

Remark: *In open-shell nuclei, the ground state is almost degenerate with respect to the excitation of pairs of nucleons in the same single-particle energy level*

State of the Art

The salient feature of the self-consistent Gorkov-Green's function approach consists in the

- Breaking of the symmetry associated with *particle-number:* $U_Z(1) \times U_N(1)$

⇝ V. Somà et al. *Phys. Rev. C* **84**, 064317 (2011)

^{44}Ca and ^{74}Ni: binding energy
^{43}Ca and ^{45}Ca: neutron addition and removal spectral distribution
^{45}Cl, ^{47}Cl and ^{49}Cl: ground and excited state energies, spectroscopic factors
$18 \leq Z \leq 24$ isotopic chains: binding energy, two neutron shell gaps, one and two-proton/neutron separation energy, charge radius
^{50}Cr, ^{52}Cr and ^{54}Cr: charge density distribution
Lepton scattering in ^{40}Ar and ^{48}Ti: neutron spectral function, charge density distr.
O, Ca and Ni isotopes: binding energy, two-neutron separation energy, charge radius
^{15}C ^{47}Ca, ^{49}Ca, ^{51}Ca, ^{55}Ca, ^{53}K and ^{55}Sc: low-lying excited states

ADC(2) with 2N forces

Phys. Rev. C **87**, 011303 (2013)
Phys. Rev. C **89**, 024323 (2014)
Phys. Rev. C **105**, 044330 (2022)

ADC(2) with 2N+3N forces

Phys. Rev. C **89**, 061301 (2014)
Phys. Rev. C ***100***, 062501 (2019)
Phys. Rev. Lett. **128**, 022502 (2022)
Eur. Phys. J A **57**, 135 (2021)
ArXiv:2302.08382
ArXiv:2310.19547

Excited-state energies, reduced EM multipole transition probabilities, γ-emission/absorption cross-sections... of even-even open-shell nuclei: ⇝ ***Gorkov's polarization propagator***

- Additional breaking *of the symmetry associated with angular momentum:*

$$U_Z(1) \times U_N(1) \times SU(2)$$

⇝ A. Scalesi's poster!

THEORETICAL FRAMEWORK

The model conveniently is formulated in second-quantization formalism.

- The **single-particle space** $\mathscr{H}_1$ is split into two blocks, characterized by the sign of the total angular mom. projection along the z axis, j_z. $\Longrightarrow$ two pairs of creation/annihilation operators:

$$a_b,\ a_{\bar{b}} \qquad a_b^\dagger,\ a_{\bar{b}}^\dagger$$

where the *involution* In s.p. space ($\rightsquigarrow$ *time reversal*) is defined:

$$a_{\bar{b}} = \eta_b a_{\tilde{b}} \qquad a_{\bar{b}}^\dagger = \eta_b a_{\tilde{b}}^\dagger$$

with

$$\tilde{b} \equiv (n, \ell, j, -m, q) \qquad b \equiv (n, \ell, j, m, q)$$

where

$$\eta_b = (-1)^{\ell - j - m} \qquad \eta_b \eta_b^* = \eta_b^2 = 1 \qquad \eta_b \eta_{\tilde{b}} = -1$$

and q $\Longrightarrow$ z-component of the isospin

- The two partitions of the single-particle space constitute the **Nambu space** (2-dimens.) Introducing the superscripts $g = 1, 2$ one groups the creation/annihilation oper. into

$$\mathbf{A}_a \equiv \begin{pmatrix} a_a \\ \bar{a}_a^\dagger \end{pmatrix} \qquad \mathbf{A}_a^\dagger = \begin{pmatrix} a_a^\dagger & \bar{a}_a \end{pmatrix}$$

and $\mathbf{A}_a^* \equiv (\mathbf{A}_a^\dagger)^T$, obeying the canonical anticommutation rules

$$\left\{ A_a^g, A_b^{g'} \right\} = \delta_{a\bar{b}} \delta_{g\bar{g}'} \qquad \left\{ A_a^g, A_b^{\dagger\, g'} \right\} = \delta_{ab} \delta_{gg'} \qquad \left\{ A_a^{\dagger\, g}, A_b^{\dagger\, g'} \right\} = \delta_{g\bar{g}'} \delta_{a\bar{b}}$$

with $\bar{g} = \begin{cases} 1 \text{ if } g = 2 \\ 2 \text{ if } g = 1 \end{cases}$ These define the elements of a *metric tensor*

involution in Nambu space $\Longrightarrow$ Nambu-Covariant Pertubation Theory in the *Appendix*

Theoretical Framework

- The system is described by the grand-canonical potential Ω, replacing the Hamiltonian, H:

$$H = T + V^{2\mathrm{N}} \quad \Longrightarrow \quad \Omega = \underbrace{T + U - \mu_p Z - \mu_n N}_{\equiv \Omega_U} + \underbrace{V^{2\mathrm{N}} - U}_{\equiv \Omega_I}$$

where

$$T = \sum_{ab} t_{ab}\, a^\dagger_a a_b \qquad \text{with} \qquad t_{ab} \equiv (a|T|b) \qquad \text{is the } \textit{kinetic energy} \text{ operator}$$

$$V^{2\mathrm{N}} = \sum_{\substack{ab\\cd}} \frac{1}{(2!)^2} \bar{v}_{abcd}\, a^\dagger_a a^\dagger_b a_d a_c \qquad \text{with} \qquad \bar{v}_{abcd} \equiv \left[(ab|V^{2\mathrm{N}}|cd) - (ab|V^{2\mathrm{N}}|dc)\right]$$

is the partially antisymmetrized two-body *potential energy* operator

and

$$U = \frac{1}{2}\sum_{ab}\left[u^{11}_{ab}\, a^\dagger_a a_b + u^{22}_{ab}\, a_{\bar{a}} a^\dagger_{\bar{b}} + u^{12}_{ab}\, a^\dagger_a a^\dagger_{\bar{b}} + u^{21}_{ab}\, a_{\bar{a}} a_b\right]$$

is a one-body *auxiliary potential*, explicitly **breaking** particle number symmetry $U(1)$.

- **Paradigm**: expansion scheme around a single reference state that builds the correlated state on top of a Bogoliubov vacuum that incorporates static pairing correlations

Physical symmetry	Group	Correlations
Particle number	$U_Z(1) \times U_N(1)$	*Pairing / superfluidity*
Rotations in 3 dim. space	$SU(2)$	*Quadrupole deformation*

The One-Body Propagator

- The Gorkov-Green's function in Nambu space and time repr. is defined as

$$i\mathbf{G}_{ab}(t,t') \equiv \langle\Psi_0|T\{\mathbf{A}_a(t)\odot\mathbf{A}_b^*(t')\}|\Psi_0\rangle$$

Since the Hamiltonian is time-independent, the FT of the one-body propagator becomes

$$\mathbf{G}_{ab}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}(t-t')\, e^{i\omega(t-t')}\mathbf{G}_{ab}(t-t')$$

Carrying out the integration, the *Lehmann representation* can be recast as

$$G_{ab}^{gg'}(\omega) = \sum_k \frac{{}^k\chi_a^g\ {}^k\chi_b^{g'*}}{\omega-(\Omega_k-\Omega_0)/\hbar+i\eta} + \sum_k \frac{{}^k\Upsilon_a^g\ {}^k\Upsilon_b^{g'*}}{\omega+(\Omega_k-\Omega_0)/\hbar-i\eta}$$

where $E_k^{(u)\pm} \equiv \mu_u \pm (\Omega_k-\Omega_0)$ with $u=p,\,n$ are the ***separation energies*** between the g.s. of the A-body system and the excited state k of the $A\pm 1$-body system.

$$E_k^{(p)\pm} \approx \pm(\langle\Psi_k^{\mathrm{SB}}|H|\Psi_k^{\mathrm{SB}}\rangle - \langle\Psi_0^{\mathrm{SB}}|H|\Psi_0^{\mathrm{SB}}\rangle) \mp \mu_p[\langle\Psi_k^{\mathrm{SB}}|Z|\Psi_k^{\mathrm{SB}}\rangle - (Z\pm 1)]$$
$$E_k^{(n)\pm} \approx \pm(\langle\Psi_k^{\mathrm{SB}}|H|\Psi_k^{\mathrm{SB}}\rangle - \langle\Psi_0^{\mathrm{SB}}|H|\Psi_0^{\mathrm{SB}}\rangle) \mp \mu_n[\langle\Psi_k^{\mathrm{SB}}|N|\Psi_k^{\mathrm{SB}}\rangle - (N\pm 1)]$$

whereas the residues of the poles are proportional to the ***spectroscopic amplitudes***

$${}^k\Upsilon_b^1 \equiv \langle\Psi_k|A_b^1|\Psi_0\rangle = \langle\Psi_k|a_b|\Psi_0\rangle \qquad {}^k\chi_b^1 \equiv \langle\Psi_0|A_b^1|\Psi_k\rangle = \langle\Psi_0|a_b|\Psi_k\rangle$$
$${}^k\Upsilon_b^2 \equiv \langle\Psi_k|A_b^2|\Psi_0\rangle = \langle\Psi_k|a_b^\dagger|\Psi_0\rangle \qquad {}^k\chi_b^2 \equiv \langle\Psi_0|A_b^2|\Psi_k\rangle = \langle\Psi_0|a_b^\dagger|\Psi_k\rangle$$

The spectroscopic amplitudes are not *independent*: $(-1)^g\ [{}^k\chi_a^g]^* = {}^k\Upsilon_{\bar{a}}^{\bar{g}}$

- Physical observables that can be evaluated from $i\mathbf{G}_{ab}(t,t')$: *see the Appendix !*

THE POLARIZATION PROPAGATOR

The construction of the Gorkov response functions recalls the Dyson case:

$$R^{gg'g''g'''}_{abcd}(t,t',t'',t''') \equiv G^{gg'g''g'''}_{abcd}(t,t',t'',t''') - G^{gg''}_{ac}(t,t'')G^{g'g'''}_{bd}(t',t''')$$

where the two-body propagator is a rank-four tensor (16 elements) in Nambu space,

$$i^2\mathbf{G}_{abcd}(t,t',t'',t''') \equiv \langle\Psi_0|T\{\mathbf{A}_a(t)\odot\mathbf{A}_b(t')\odot\mathbf{A}^*_d(t''')\odot\mathbf{A}^*_c(t'')\}|\Psi_0\rangle$$

with the convention by J. Schirmer, *Phys. Rev. A* **26**, 5, 2395-2416 (1982)

- Switching to the two-time limit the Gorkov *polarization propagator* is obtained:

$$\Pi^{gg''g'''g'}_{acdb}(t,t') \equiv \lim_{\substack{t''\to t^+ \\ t'''\to t'^+}} R^{gg'g''g'''}_{abcd}(t,t',t'',t''')$$

It has 10 *anomalous* and 6 *normal* components: '1111', '1212', '2121', '1221', '2112' and '2222'.

Explicitly:

$$\Pi^{gg''g'''g'}_{acdb}(t,t') = -i\langle\Psi^A_0|T\Big\{A^g_a(t)A^{g'}_b(t')A^{\dagger\, g'''}_d(t'^+)A^{\dagger\, g''}_c(t^+)\Big\}|\Psi^A_0\rangle$$

$$+i\langle\Psi^A_0|T\Big\{A^g_a(t)A^{\dagger\, g''}_c(t^+)\Big\}|\Psi^A_0\rangle\langle\Psi^A_0|T\Big\{A^{g'}_b(t')A^{\dagger\, g'''}_d(t'^+)\Big\}|\Psi^A_0\rangle$$

Analogously, the Fourier Transform of the polarization propagator yields

$$\Pi^{gg''g'''g'}_{acdb}(\omega) \equiv \int_{-\infty}^{+\infty} \mathrm{d}(t-t')e^{i\omega(t-t')}\Pi^{gg''g'''g'}_{acdb}(t-t')$$

and fulfills the following *symmetry property* under complex conjugation:

$$\Pi^{gg''g'''g'}_{acdb}(\omega) = (-1)^{\bar{g}+\bar{g}'+\bar{g}''+\bar{g}'''}\ [\Pi^{\bar{g}\bar{g}''\bar{g}'''\bar{g}'}_{\bar{a}\bar{c}\bar{d}\bar{b}}(-\omega)]^*$$

- The Lehmann representation of the Gorkov polarization propagator gives

$$\Pi^{gg''g'''g}_{acdb}(\omega) \equiv \Pi^{+\,gg''g'''g}_{acdb}(\omega) + \Pi^{-\,gg''g'''g}_{acdb}(\omega)$$

The two contributions contain the same information and are again related by complex conjugation

$$\Pi^{+\,gg''g'''g'}_{acdb}(\omega) = (-1)^{\bar{g}+\bar{g}'+\bar{g}''+\bar{g}'''}[\Pi^{+\,\bar{g}\bar{g}''\bar{g}'''\bar{g}'}_{\bar{a}\bar{c}\bar{d}\bar{b}}(-\omega)]^*$$

where the l.h.s. (r.h.s.) is analytical in the upper (lower) part of the complex plane for ω,

$$\Pi^{+\,gg''g'''g'}_{acdb}(\omega) = \sum_{k\neq 0} \frac{{}^k\chi^{gg''}_{ac}\;{}^k\chi^{*\,g'''g'}_{db}}{\omega - (\Omega_k - \Omega_0)/\hbar + i\eta} \qquad \Pi^{-\,gg''g'''g'}_{acdb}(\omega) = -\sum_{k\neq 0} \frac{{}^k\Upsilon^{gg''}_{ac}\;{}^k\Upsilon^{*\,g'''g'}_{db}}{\omega + (\Omega_k - \Omega_0)/\hbar - i\eta}$$

the poles, for $U(1)$-SB states, approx. coincide with the energy of the excited states of the A-body system with respect to the g.s. energy $E_k \approx \Omega_k - \Omega_0$ and the transition matrix elements, fulfilling

$$-(-1)^{g+g'}\;{}^k\Upsilon^{gg'}_{ab} = [{}^k\chi^{\bar{g}\bar{g}'}_{\bar{a}\bar{b}}]^*$$

have been defined and orthogonality between the A-body states has been exploited. Explicitly

$$\begin{aligned}
{}^k\chi^{22}_{bc} &\equiv \langle\Psi_0|A^2_b A^{\dagger\,2}_c|\Psi_k\rangle = \langle\Psi_0|a^\dagger_{\bar{b}} a_{\bar{c}}|\Psi_k\rangle &\quad {}^k\Upsilon^{22}_{bc} &\equiv \langle\Psi_k|A^2_b A^{\dagger\,2}_c|\Psi_0\rangle = \langle\Psi_k|a^\dagger_{\bar{b}} a_{\bar{c}}|\Psi_0\rangle \\
{}^k\chi^{12}_{bc} &\equiv \langle\Psi_0|A^1_b A^{\dagger\,2}_c|\Psi_k\rangle = \langle\Psi_0|a_b a_{\bar{c}}|\Psi_k\rangle &\quad {}^k\Upsilon^{12}_{bc} &\equiv \langle\Psi_k|A^1_b A^{\dagger\,2}_c|\Psi_0\rangle = \langle\Psi_k|a_b a_{\bar{c}}|\Psi_0\rangle \\
{}^k\chi^{11}_{bc} &\equiv \langle\Psi_0|A^1_b A^{\dagger\,1}_c|\Psi_k\rangle = \langle\Psi_0|a_b a^\dagger_c|\Psi_k\rangle &\quad {}^k\Upsilon^{21}_{bc} &\equiv \langle\Psi_k|A^2_b A^{\dagger\,1}_c|\Psi_0\rangle = \langle\Psi_k|a^\dagger_{\bar{b}} a^\dagger_c|\Psi_0\rangle \\
{}^k\chi^{21}_{bc} &\equiv \langle\Psi_0|A^2_b A^{\dagger\,1}_c|\Psi_k\rangle = \langle\Psi_0|a^\dagger_{\bar{b}} a^\dagger_c|\Psi_k\rangle &\quad {}^k\Upsilon^{11}_{bc} &\equiv \langle\Psi_k|A^1_b A^{\dagger\,1}_c|\Psi_0\rangle = \langle\Psi_k|a_b a^\dagger_c|\Psi_0\rangle
\end{aligned}$$

as in the one-body GF case, the anomalous elements *vanish* between $U(1)$-conserving states.

- For a general one-body operator that mediates the transition between two the A-body states

$$\langle\Psi_p|\mathcal{O}|\Psi_0\rangle = \sum_{ab}(a|\mathcal{O}|b)\langle\Psi_p|a_b^\dagger a_a|\Psi_0\rangle$$

  - **Example**: reduced *electric* (R=E) and *magnetic* (R=M) multipole transition probabilities between states with angular momentum J_0 and J_p

$$B(J_0 \to J_p, R\ell) \equiv \frac{1}{2J_0+1}\sum_{M_0}\sum_{M_p}\sum_{m}|\langle\Psi_p|\mathfrak{Q}_{\ell m}(R)|\Psi_0\rangle|^2$$

where $\mathfrak{Q}_{\ell m}(R)$ are the transition operators with angular momentum ℓ and projection m

$$\langle\Psi_p|\mathfrak{Q}_{\ell m}(R)|\Psi_0\rangle = \sum_{ab}(a|\mathfrak{Q}_{\ell m}(R)|b)\langle\Psi_p|[A_a^{1\,\dagger}\otimes A_b^1]^\ell_m|\Psi_0\rangle$$

which are expressed in terms of the angular-momentum-coupled transition matrix elements

$$[A_a^{\dagger\,1}\otimes A_b^1]^\ell_m = [a_a^\dagger \otimes a_b]^\ell_m = \sum_{m_a m_b}(j_a j_b \ell|m_a - m_b m)(-1)^{-m_b}a_a^\dagger a_b$$

and the matrix elements between the s.p. states and the EM mult. transition oper. are given by

$$(a|\mathfrak{Q}_{\ell m}(E)|b) = \int \mathrm{d}^3r\ (a|r^\ell Y_\ell^m(\theta,\phi)\rho(\mathbf{r})|b)$$

$$(a|\mathfrak{Q}_{\ell m}(M)|b) = \int \mathrm{d}^3r\ (a|\mathbf{j}(\mathbf{r})\cdot\mathbf{L}r^\ell Y_\ell^m(\theta,\phi)|b)$$

where $\rho(\mathbf{r}) = e\delta(\mathbf{r}-\mathbf{r}')$ and $\mathbf{j}(\mathbf{r}) = \frac{\hbar}{2mi}[\delta(\mathbf{r}-\mathbf{r}')\overrightarrow{\nabla}' - \overleftarrow{\nabla}'\delta(\mathbf{r}-\mathbf{r}')]$ (pointlike charge distrib.)

Perturbative Expansion

of the polarization propagator

- Let us consider the *perturbative expansion* of Gorkov's polarization propagator in terms of $\Omega_I = V^{2N} - U$ with the implied second-quantization operators the interaction picture:

⇝ J. Schirmer, *Phys. Rev. A* ***26***, 5, 2395-2416 (1982)

$$
\begin{aligned}
\Pi^{g_1g_2g_3g_4}_{acdb}(t,t^+,t'^+,t') &= -i\sum_{l=0}^{+\infty}\left(\frac{-i}{\hbar}\right)^l\frac{1}{l!}\int_{-\infty}^{+\infty}\mathrm{d}t_1\ldots\int_{-\infty}^{+\infty}\mathrm{d}t_l\,\overbrace{\langle\Phi_0|T\Big\{\Omega_I(t_1)\ldots\Omega_I(t_l)A_{I\,a}^{g_1}(t)A_{I\,b}^{g_2}(t')A_{I\,d}^{\dagger g_4}(t'^+)A_{I\,c}^{\dagger g_3}(t^+)\Big\}|\Phi_0\rangle_C}^{\text{(P)}} \\
&+ i\left[\sum_{m=0}\left(\frac{-i}{\hbar}\right)^m\frac{1}{m!}\int_{-\infty}^{+\infty}\mathrm{d}t_1\ldots\int_{-\infty}^{+\infty}\mathrm{d}t_m\langle\Phi_0|T\Big\{\Omega_I(t_1)\ldots\Omega_I(t_m)A_{I\,a}^{g_1}(t)A_{I\,c}^{\dagger g_3}(t^+)\Big\}|\Phi_0\rangle_C\right] \\
&\times\left[\sum_{n=0}\left(\frac{-i}{\hbar}\right)^n\frac{1}{n!}\int_{-\infty}^{+\infty}\mathrm{d}t_1\ldots\int_{-\infty}^{+\infty}\mathrm{d}t_n\,\underbrace{\langle\Phi_0|}\,T\Big\{\Omega_I(t_1)\ldots\Omega_I(t_n)A_{I\,b}^{g_2}(t')A_{I\,d}^{\dagger g_4}(t'^+)\Big\}|\Phi_0\rangle_C\right]
\end{aligned}
$$

where unperturbed reference state — connected contributions only!

- Time ordered products in (P) are evaluated by means of *Wick's theorem*, converting them into fully-contracted normal-ordered products of second-quantization operators.

Caveat: contractions between two creation and annihilation operators do not vanish!

Example: conventions for non-canonical contractions, valid for all but *Bogoliubov* contributions

Contr. / Indic.	$a_e a_f$	$a_{\bar e} a_{\bar f}$	$a^\dagger_e a^\dagger_f$	$a^\dagger_{\bar e} a^\dagger_{\bar f}$	$a_e a^\dagger_{\bar f}$	$a^\dagger_e a_{\bar f}$
e, f internal	$iG^{(0)\,12}_{ef}$ $f \mapsto \bar f$	$iG^{(0)\,12}_{ef}$ $\bar e \mapsto e$	$iG^{(0)\,21}_{ef}$ $e \mapsto \bar e$	$iG^{(0)\,21}_{ef}$ $\bar f \mapsto f$	$-iG^{(0)\,22}_{fe}$ $e \mapsto \bar e$	$iG^{(0)\,22}_{ef}$ $e \mapsto \bar e$
e external, f internal	$iG^{(0)\,12}_{ef}$ $f \mapsto \bar f$	$-iG^{(0)\,12}_{fe}$ $\bar f \mapsto f$	$-iG^{(0)\,21}_{fe}$ $f \mapsto \bar f$	$iG^{(0)\,21}_{ef}$ $\bar f \mapsto f$	$iG^{(0)\,11}_{ef}$ $\bar f \mapsto f$	$-iG^{(0)\,11}_{fe}$ $\bar f \mapsto f$
e internal, f external	$-iG^{(0)\,12}_{fe}$ $e \mapsto \bar e$	$iG^{(0)\,12}_{ef}$ $\bar e \mapsto e$	$iG^{(0)\,21}_{ef}$ $e \mapsto \bar e$	$-iG^{(0)\,21}_{fe}$ $\bar e \mapsto e$	$-iG^{(0)\,22}_{fe}$ $e \mapsto \bar e$	$iG^{(0)\,22}_{ef}$ $e \mapsto \bar e$
e, f external	–	–	–	–	–	–

DIAGRAMMATIC REPRESENTATION

of the polarization propagator

- Graphical interpretation of fully-contracted Wick's-theorem contributions in terms of *Feynman diagrams* for the polarization propagator in *time representation.* The conventions below hold:

  - **One-body vertices**: four inequivalent types

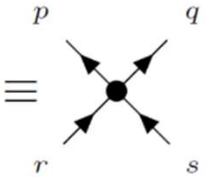
$\frac{1}{2!}u^{11}_{ef} \equiv$ 

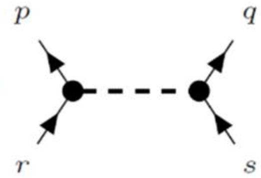
$\frac{1}{2!}u^{12}_{ef} \equiv$ 

$\frac{1}{2!}u^{21}_{ef} \equiv$ 

$\frac{1}{2!}u^{22}_{ef} \equiv$ 

  - **Two-body vertex**: two notations

$\frac{1}{2!^2}\bar{v}_{pqrs} \equiv$ 

$\equiv$ 

(i) Abrikosov – Hügenholtz (ii) Bloch-Brandow

- Feynman diagrams of order $l=n+m$ are graphs with m (n) one-body (two-body) vertices linked one another by $2n+m+2$ unperturbed one-body propagators. In the latter, the orientation of the arrows depend on the Nambu indices.

  - Unperturbed **one-body propagators**:
    2 anomalous: '21' & '12'
    and *2 normal*: '11' & '22'

$G^{(0)\,21}_{ab}(t,t') \equiv$ 

$G^{(0)\,12}_{ab}(t,t') \equiv$ 

$G^{(0)\,22}_{ab}(t,t') \equiv$ 

$G^{(0)\,11}_{ab}(t,t') \equiv$ 

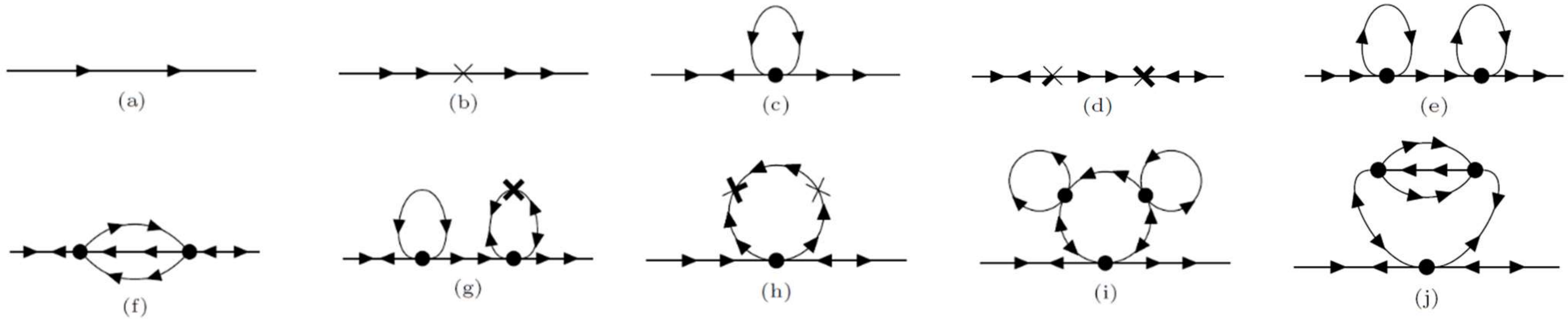
  - *External* single-particle indices

	$g_3 \rightarrow$	1	1	2	2
$g_1 \downarrow$	$g_2 \downarrow$ \ $g_4 \rightarrow$	1	2	1	2
1	1	$a\ c$ $d\ b$	$a\ c$ $\bar{d}\ b$	$a\ \bar{c}$ $d\ b$	$a\ \bar{c}$ $\bar{d}\ b$
1	2	$a\ c$ $d\ \bar{b}$	$a\ c$ $\bar{d}\ \bar{b}$	$a\ \bar{c}$ $d\ \bar{b}$	$a\ \bar{c}$ $\bar{d}\ \bar{b}$
2	1	$\bar{a}\ c$ $d\ b$	$\bar{a}\ c$ $\bar{d}\ b$	$\bar{a}\ \bar{c}$ $d\ b$	$\bar{a}\ \bar{c}$ $\bar{d}\ b$
2	2	$\bar{a}\ c$ $d\ \bar{b}$	$\bar{a}\ c$ $\bar{d}\ \bar{b}$	$\bar{a}\ \bar{c}$ $d\ \bar{b}$	$\bar{a}\ \bar{c}$ $\bar{d}\ \bar{b}$

Diagrammatic Categories

of the polarization propagator

- A *dressing* of order p in a Feynman diagram of order $l \geq p$ is a sub-graph with connected vertices which can be isolated by cutting two propagation lines. **Examples** for the one-body prop. :



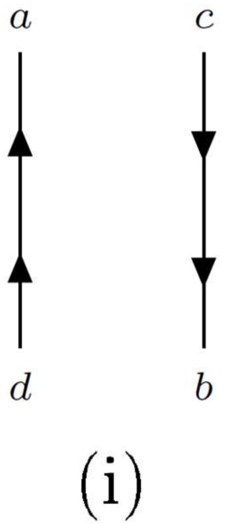
To specify the topology of a graph, the *orientation* of all propagation lines must be specified. A non-oriented Feynman diagram is called a *tree*.

- The application of Wick's theorem to the term (P) of the perturbation expansion gives rise to contributions which can be classified according to their topology into **5** categories:

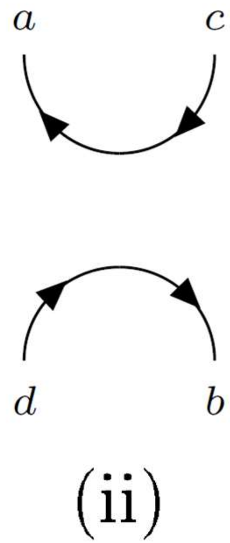
Example:

$\Pi^{1111}_{acdb}(t,t')$

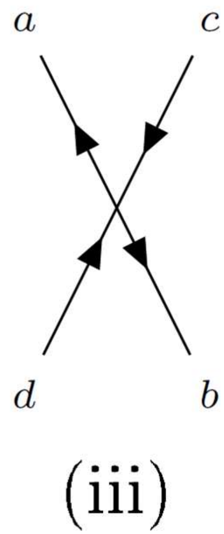
- *Disjoint diagrams of **direct** type* →



- *Disjoint diagrams of **exchange** type* →



- *Disjoint diagrams of **Bogoliubov** type* →

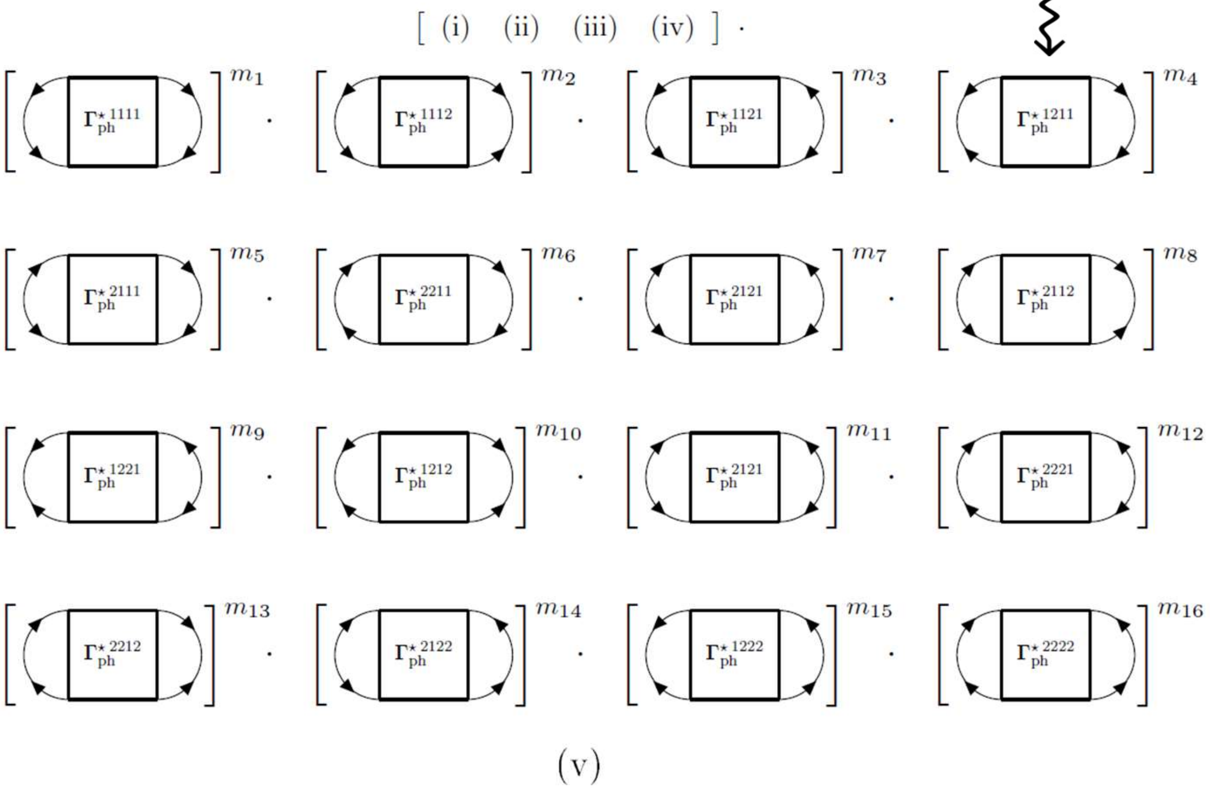


- *Composite* diagrams contain at least one vertex that can be reabsorbed as a dressing in one unperturbed one-body propagators. Otherwise, the diagram is of *skeleton* type.

Example: $\Pi^{1111}_{acdb}(t, t')$

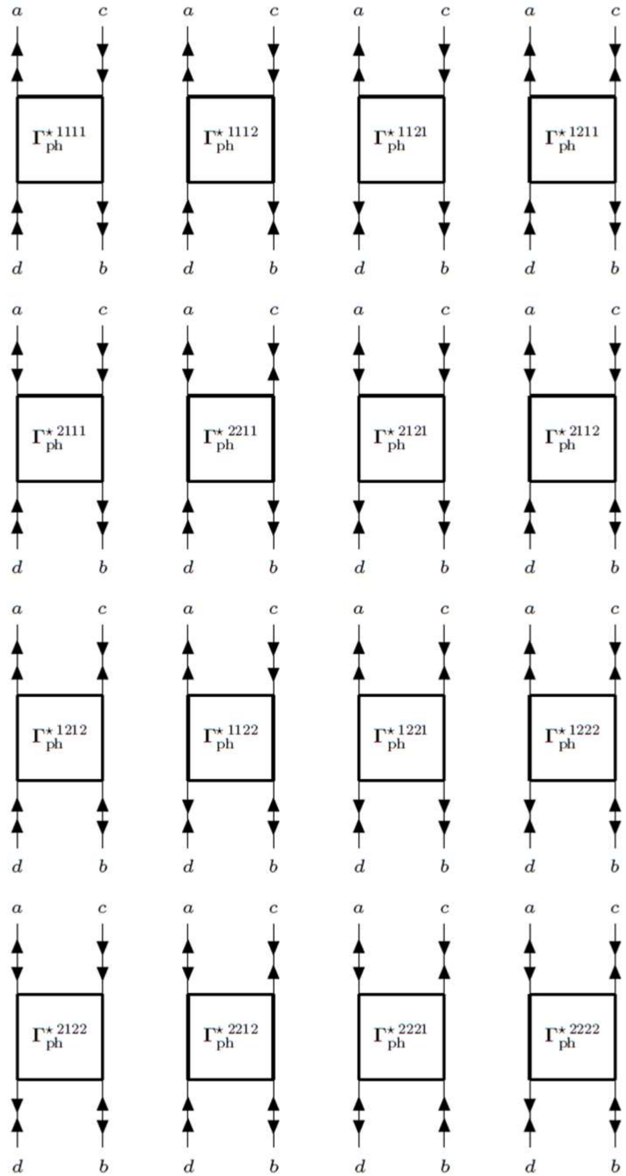
Conjoint *diagrams* →

proper particle-hole vertex



(v)

Disconnected *diagrams* ↑



(iv)

Gorkov's Bethe-Salpeter Equations

- Gorkov's polarization propagator has proven to fulfill the following *self-consistent* equations

$$\Pi^{gg''g'''g'}_{acdb}(t,t^+,t'^+,t') = \underbrace{\Pi^{\mathrm{D}\,gg''g'''g'}_{acdb}(t,t^+,t'^+,t')} + \underbrace{\Pi^{\mathrm{B}\,gg''g'''g'}_{acdb}(t,t^+,t'^+,t')} + \frac{1}{\hbar}\sum_{\substack{g_1g_2\\g_3g_4}}\sum_{\substack{ef\\gh}}\int_{-\infty}^{+\infty} ds_1$$

$$\cdot \int_{-\infty}^{+\infty} ds_2 \int_{-\infty}^{+\infty} ds_3 \int_{-\infty}^{+\infty} ds_4\, \underbrace{\Pi^{\mathrm{D}\,g_2g_1g'''g'}_{fedb}(s_2,s_1,t^+,t)}\, \underbrace{\Gamma^{\star\,g_1g_2g_3g_4}_{\mathrm{ph}\,efgh}(s_1,s_2,s_3,s_4)}\, \overbrace{\Pi^{gg''g_4g_3}_{achg}(t',t'^+,s_4,s_3)}$$

disj. direct pol. propagator; disj. Bogoliubov pol. propagator; three-time pol. propagator; disjoint direct three-time pol. propagator

where

$$\Sigma^{\phi gg''}_{ad}(t,t'') = -i\sum_{bcef}\sum_{g'}(\bar{v}_{acef}\delta_{1g}+\bar{v}_{c\bar{e}\bar{a}f}\delta_{g2})\int_{-\infty}^{+\infty} dt'\, G^{\phi g1g'1}_{efbc}(t,t,t',t^+)\,G^{\phi-1g'g''}_{bd}(t',t'') \quad \Leftarrow \quad \Gamma^{\star\,g_1g_2g_3g_4}_{\mathrm{ph}\,efgh}(s_1,s_2,s_3,s_4) = \left.\frac{\delta\Sigma^{\phi g_1g_2}_{ef}(s_1,s_2)}{\delta G^{g_3g_4}_{gh}(s_3,s_4)}\right|_{\phi(t)=0}$$

self-energy, in terms of the two-body propagator — *proper particle-hole vertex*

Where the two and three-time polarization propagator of direct and Bogoliubov types are special limits of the four-time polarization propagator ($\propto R^{g_1g_2g_3g_4}_{abcd}(t_1,t_2,t_3,t_4)$):

$$\Pi^{\mathrm{D}\,g_4g_3g_2g_1}_{acdb}(t_1,t_2,t_3,t_4) = -iG^{g_1g_4}_{ad}(t_1,t_4)G^{g_2g_3}_{bc}(t_2,t_3) \qquad \Pi^{\mathrm{B}\,g_4g_3g_2g_1}_{acdb}(t_1,t_2,t_3,t_4) = iG^{\bar{g}_4g_3}_{\bar{d}c}(t_4,t_3)G^{g_2\bar{g}_3}_{a\bar{b}}(t_1,t_2)$$

- In energy representation, Gorkov's Bethe-Salpeter equations (GBSE) become

$$\Pi^{gg''g'''g'}_{acdb}(\omega) = \Pi^{\mathrm{D}\,gg''g'''g'}_{acdb}(\omega) + \Pi^{\mathrm{B}\,gg''g'''g'}_{acdb}(\omega) + \frac{1}{\hbar}\sum_{efgh}\sum_{\substack{g_1g_2\\g_3g_4}}\int_{-\infty}^{+\infty}\frac{d\Omega_1}{(2\pi)}\int_{-\infty}^{+\infty}\frac{d\Omega_2}{(2\pi)}\Pi^{\mathrm{D}\,g_2g_1g'''g'}_{fedb}\left(\tfrac{\omega+\Omega_1}{2},\tfrac{\omega-\Omega_1}{2}\right)$$

$$\cdot\Gamma^{\star\,g_1g_2g_3g_4}_{\mathrm{ph}\,efgh}\left(\tfrac{\omega-\Omega_1}{2},\omega,\omega-2\Omega_2\right)\Pi^{gg''g_4g_3}_{achg}(2\Omega_2,\omega-2\Omega_2)\ .$$

In contrast with Gorkov's equations, in energy repr. it remains an *integral* equation!

Gorkov's Bethe-Salpeter Equations

- The *proper* particle-hole vertex contains 2-body vertex insertions corresponding graphically to **conjoint skeleton** polarization propagator diagrams with amputated external legs.

Zeroth-order terms do not contribute by definition: $\Gamma^{\star\,(0)\,g_1g_2g_3g_4}_{\mathrm{ph}\quad\, efgh} = 0$

One-body vertices are excluded, as they contribute only to the *improper* p-h vertex: $\Gamma_{\mathrm{ph}\, efgh}^{\;g_1g_2g_3g_4}$

- The explicit calculation of the first-order contributions to $\Gamma^{\star\;g_1g_2g_3g_4}_{\mathrm{ph}\; efgh}$ yield:

$$\Gamma^{\star\,(1)\,1111}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \delta(s_1-s_2)\delta(s_2-s_3)\delta(s_3-s_4)\bar{v}_{hegf}$$

$$\Gamma^{\star\,(1)\,1212}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \delta(s_1-s_2)\delta(s_2-s_3)\delta(s_3-s_4)\bar{v}_{\bar{f}e\bar{h}g}$$

$$\Gamma^{\star\,(1)\,1122}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \delta(s_1-s_2)\delta(s_2-s_3)\delta(s_3-s_4)\bar{v}_{\bar{g}ef\bar{h}}$$

$$\Gamma^{\star\,(1)\,2211}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \delta(s_1-s_2)\delta(s_2-s_3)\delta(s_3-s_4)\bar{v}_{h\bar{f}\bar{e}g}$$

$$\Gamma^{\star\,(1)\,2121}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \delta(s_1-s_2)\delta(s_2-s_3)\delta(s_3-s_4)\bar{v}_{h\bar{g}f\bar{e}}$$

$$\Gamma^{\star\,(1)\,2222}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \delta(s_1-s_2)\delta(s_2-s_3)\delta(s_3-s_4)\bar{v}_{\bar{f}\bar{g}\bar{e}\bar{h}}$$

i.e. the *normal* components of $\mathbf{\Gamma}^{\star}_{\mathrm{ph}\;efgh}$. The *anomalous* components vanish at first-order:

$$\Gamma^{\star\,(1)\,1112}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \Gamma^{\star\,(1)\,1121}_{ph\quad\, efgh}(s_1,s_2,s_3,s_4) = \Gamma^{\star\,(1)\,1211}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \Gamma^{\star\,(1)\,2111}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = 0$$

$$\Gamma^{\star\,(1)\,1222}_{\mathrm{ph}\quad\, efgh}(t_1,t_2,t_3,t_4) = \Gamma^{\star\,(1)\,2122}_{\mathrm{ph}\quad\, efgh}(t_1,t_2,t_3,t_4) = \Gamma^{\star\,(1)\,2212}_{\mathrm{ph}\quad\, efgh}(t_1,t_2,t_3,t_4) = \Gamma^{\star\,(1)\,2221}_{\mathrm{ph}\quad\, efgh}(t_1,t_2,t_3,t_4) = 0$$

$$\Gamma^{\star\,(1)\,1221}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = \Gamma^{\star\,(1)\,2112}_{\mathrm{ph}\quad\, efgh}(s_1,s_2,s_3,s_4) = 0$$

Automated Implementation of Wick's Theorem

for Gorkov's polarization propagator

A code implementing Wick's theorem in the term (P) of the expansion formula for Gorkov's polarization propagator has been developed in **Mathematica** and Jupyter up to third order.

At order $l = m{+}n$ with m (n) one-body (two-body) vertices there are $4n{+}2m\text{-}3!!$ contributions

Generation Process

Key: encode fully-contracted contributions into 2-dim. arrays of integers (≡ rectang. matrices)

- Example: *third order*

$$\underbrace{V^{2N}(t_1)}_{p\,q\,s\,r \;\to\; 1\,2\,3\,4}\quad \underbrace{U(t_2)}_{e\,f \;\to\; 5\,6}\quad \underbrace{U(t_3)}_{g\,h \;\to\; 7\,8}$$

$$\underbrace{V^{2N}(t_1)}_{p\,q\,s\,r \;\to\; 1\,2\,3\,4}\quad \underbrace{V^{2N}(t_2)}_{t\,u\,w\,v \;\to\; 5\,6\,7\,8}\quad \underbrace{U(t_3)}_{e\,f \;\to\; 9\,10}$$

$$\underbrace{U(t_1)}_{e\,f \;\to\; 1\,2}\quad \underbrace{U(t_2)}_{g\,h \;\to\; 3\,4}\quad \underbrace{U(t_3)}_{i\,j \;\to\; 5\,6}$$

$$\underbrace{V^{2N}(t_1)}_{p\,q\,s\,r \;\to\; 1\,2\,3\,4}\quad \underbrace{V^{2N}(t_2)}_{t\,u\,w\,v \;\to\; 5\,6\,7\,8}\quad \underbrace{V^{2N}(t_3)}_{k\,l\,n\,m \;\to\; 9\,10\,11\,12}$$

▲ one and two-body vertices

$A_a^{g_1}(t)$ ↓ a	$A_b^{g_2}(t')$ ↓ b	$A_d^{\dagger\, g_4}(t')$ ↓ c	$A_c^{\dagger\, g_3}(t)$ ↓ d	Ord.
1	2	3	4	0^{th}
5	6	7	8	1^{st}
9	10	11	12	2^{nd}
13	14	15	16	3^{rd}

▲ external legs

the s.-p. indices of 2nd-quantization operators contracted together are stored in the same *row*.

All contrib. are generated by means of transp. from the *canonical* sequence (1st elem. of `Acomb`)

- Example: *third order* `Acomb[[1]] = {{1,2},{3,4},{5,6},{7,8},{9,10},{11,12},{13,14},{15,16}}`

$$a_p^\dagger(t_1)a_q^\dagger(t_1)a_s(t_1)a_r(t_1)a_t^\dagger(t_2)a_u^\dagger(t_2)a_w(t_2)a_v(t_2)\cdot a_k^\dagger(t_3)a_l^\dagger(t_3)a_n(t_3)a_m(t_3)a_a^\dagger(t)a_b(t')a_d^\dagger(t')a_c^\dagger(t) \longmapsto$$

`Acomb[[d_1]] = {{1,4},{5,16},{6,8},{11,12},{9,13},{2,14},{3,7},{10,15}}`

Automated Implementation of Wick's Theorem

for Gorkov's polarization propagator in the ADC scheme

- Next, elementary *topological rules* are exploited in order to separate the Wick's-theorem contributions according to the diagram *category* and *type* they are associated with.

	Order $l = m + n$ →	0	1		2			3			
Category	Type \ (m,n) →	$(0,0)$	$(1,0)$	$(0,1)$	$(2,0)$	$(1,1)$	$(0,2)$	$(3,0)$	$(2,1)$	$(1,2)$	$(0,3)$
Conjoint	Skeleton	0	0	24	0	0	1,728	0	0	0	311,040
	Composite	0	0	0	0	768	2,304	0	19,200	193,536	718,848
Disjoint Direct	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Left or Right-Dr.	0	16	24	160	576	1,536	1,920	11,520	61,440	228,096
	Left and Right-Dr.	0	0	0	80	192	288	1920	7,680	26,112	55,296
Disjoint Bogoliubov	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Diag. or Antidiag.-Dr.	0	16	24	160	576	1,536	1,920	11,520	61,440	228,096
	Diag. and Antidiag.-Dr.	0	0	0	80	192	288	1920	7,680	26,112	55,296
Relevant		2	32	72	480	2,304	7,680	7,680	57,600	368,640	1,596,672
Disjoint Exchange	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Above or Below-Dr.	0	16	24	160	576	1,536	1,920	11,520	61,440	228,096
	Above and Below-Dr.	0	0	0	80	192	288	1920	7,680	26,112	55,296
Disconnected		0	12	9	330	708	891	7,380	27,150	84,348	146,961
Total		3	60	105	1,050	3,780	10,395	18,900	103,950	540,540	2,027,025

At third order, there are **2,690,415** fully-contracted terms in total!

Filtration Process (jupyter, NVIDIA CUDA)

- The *equivalent* Wick's-theorem contrib. are identified and their multiplicity is stored, thanks to:
  - ⇝ the exchange symmetry of identical one and two-body vertices (*external* s.-p. index permutations)
    - ⇝ the partial antisymmetry of the two-body vertex, (*internal* s.-p. index permutations)

Category	Order $l = m + n$ → / Type \ (m,n) →	0: $(0,0)$	1: $(1,0)$	1: $(0,1)$	2: $(2,0)$	2: $(1,1)$	2: $(0,2)$	3: $(3,0)$	3: $(2,1)$	3: $(1,2)$	3: $(0,3)$
Conjoint	Skeleton	0	0	6	0	0	60	0	0	0	924
	Composite	0	0	0	0	192	96	0	3,840	7,200	2,880
Disjoint Direct	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Left or Right-Dr.	0	16	8	128	176	76	1,024	2,704	2,720	1,032
	Left and Right-Dr.	0	0	0	64	64	16	1,024	1,920	1,312	304
Disjoint Bogoliubov	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Diag. or Antidiag.-Dr.	0	16	8	128	176	76	1,024	2,704	2,720	1,032
	Diag. and Antidiag.-Dr.	0	0	0	64	64	16	1,024	1,920	1,312	304
Relevant		2	32	22	384	672	340	4,096	13,088	15,264	6,476
Disjoint Exchange	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Above or Below-Dr.	0	16	8	128	176	76	1,024	2,704	2,720	1,032
	Above and Below-Dr.	0	0	0	64	64	16	1,024	1,920	1,312	304
Disconnected		0	12	6	282	288	99	4,308	7,788	5,604	1,524
Total		3	60	36	858	1200	531	10,452	25,500	24,900	9,336

Automated Implementation of Wick's Theorem

for Gorkov's polarization propagator in the ADC scheme

- The *equivalent* Wick's-theorem contrib. are identified and their multiplicity is stored, thanks to:
  - ⇝ the exchange symmetry of identical one and two-body vertices (*external* s.-p. index permutations)
    - ⇝ the partial antisymmetry of the two-body vertex, (*internal* s.-p. index permutations)

	Order $l = m + n$ →	0	1		2			3			
Category	(m, n) → / Type	$(0,0)$	$(1,0)$	$(0,1)$	$(2,0)$	$(1,1)$	$(0,2)$	$(3,0)$	$(2,1)$	$(1,2)$	$(0,3)$
Conjoint	Skeleton	0	0	6	0	0	60	0	0	0	924
	Composite	0	0	0	0	192	96	0	3,840	7,200	2,880
Disjoint Direct	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Left or Right-Dr.	0	16	8	128	176	76	1,024	2,704	2,720	1,032
	Left and Right-Dr.	0	0	0	64	64	16	1,024	1,920	1,312	304
Disjoint Bogoliubov	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Diag. or Antidiag.-Dr.	0	16	8	128	176	76	1,024	2,704	2,720	1,032
	Diag. and Antidiag.-Dr.	0	0	0	64	64	16	1,024	1,920	1,312	304
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Disjoint Exchange	Non-Dressed	1	0	0	0	0	0	0	0	0	0
	Above or Below-Dr.	0	16	8	128	176	76	1,024	2,704	2,720	1,032
	Above and Below-Dr.	0	0	0	64	64	16	1,024	1,920	1,312	304
Disconnected		0	12	6	282	288	99	4,308	7,788	5,604	1,524
Total		3	60	36	858	1200	531	10,452	25,500	24,900	9,336

At third order, there are **70,188** inequivalent fully-contracted terms in total!

AUTOMATED IMPLEMENTATION OF WICK'S THEOREM

for Gorkov's polarization propagator in the ADC scheme

EVALUATION PROCESS

The inequivalent Wick's-theorem contributions are converted into *analytical* expressions.
The latter are in one-to-one correspondence with the amplitudes of the *Feynman graphs* in time representation.

▶ The procedure entails:

- ✓ the conversion of contractions into *one-body Gorkov-Green's functions*;
- ✓ the writing of the *summation* and *integration signs*;
- ✓ the writing of *multiplication factors* (interaction matrix elements, multiplicity, ...)
- ✓ the determination of the *sign*;

t' $\downarrow$	t_3 $\downarrow$	t_2 $\downarrow$	t_1 $\downarrow$	t $\downarrow$	ORD.
1	–	–	–	2	0^{th}
1	–	–	2	3	1^{st}
1	–	2	3	4	2^{nd}
1	2	3	4	5	3^{rd}

convention for the encoding of the time indices of one.body Gorkov-Green's functions.

convention for the conversion of *canonical* contractions for Wick's-theorem contributions of Bogoliubov type

INDIC. \ CONTR.	$a_e a_{\bar f}$	$a^\dagger_{\bar e} a^\dagger_f$	$a_e a^\dagger_f$	$a^\dagger_{\bar e} a_{\bar f}$
$e,\ f$ INTERNAL	$iG^{(0)\ 12}_{ef}$	$iG^{(0)\ 21}_{ef}$	$iG^{(0)\ 11}_{ef}$	$iG^{(0)\ 22}_{ef}$
$e = a, c$ EXT. f INTERNAL	$iG^{(0)\ 12}_{ef}$	$iG^{(0)\ 21}_{ef}$	$iG^{(0)\ 11}_{ef}$	$iG^{(0)\ 22}_{ef}$
$e = b, d$ EXT. f INTERNAL	$iG^{(0)\ 12}_{f\bar e}$ $\bar f \mapsto f$	$-iG^{(0)\ 21}_{f\bar e}$ $f \mapsto \bar f$	$iG^{(0)\ 22}_{f\bar e}$ $f \mapsto \bar f$	$-iG^{(0)\ 11}_{f\bar e}$ $\bar f \mapsto f$
e INTERNAL $f = a, c$ EXT.	$iG^{(0)\ 12}_{ef}$	$iG^{(0)\ 21}_{ef}$	$iG^{(0)\ 11}_{ef}$	$iG^{(0)\ 22}_{ef}$
e INTERNAL $f = b, d$ EXT.	$-iG^{(0)\ 12}_{\bar f e}$ $e \mapsto \bar e$	$iG^{(0)\ 21}_{\bar f e}$ $\bar e \mapsto e$	$iG^{(0)\ 22}_{\bar f e}$ $e \mapsto \bar e$	$-iG^{(0)\ 11}_{\bar f e}$ $\bar e \mapsto e$
$e,\ f$ EXTERNAL	–	–	–	–

Output: amplitudes of *conjoint-composite* Feynman diagrams of Gorkov's polarization propagator at third order with $(m,n) = (1,2)$ and Nambu component '*1111*'

```
In[•]:= For[u = 1, u ≤ 1800, u++,
  If [GGFNambu⟦u, 1, 1⟧ == 1 && GGFNambu⟦u, 1, 2⟧ == 1 && GGFNambu⟦u, 2, 1⟧ == 1 && GGFNambu⟦u, 2, 2⟧ == 1 && GGFNambu⟦u, 3, 1⟧ == 1 && GGFNambu⟦u, 3, 2⟧ == 1 &&
    GGFNambu⟦u, 4, 1⟧ == 1 && GGFNambu⟦u, 4, 2⟧ == 1 && GGFNambu⟦u, 5, 1⟧ == 1 && GGFNambu⟦u, 5, 2⟧ == 1 && GGFNambu⟦u, 6, 1⟧ == 1 && GGFNambu⟦u, 6, 2⟧ == 1 &&
    GGFNambu⟦u, 7, 1⟧ == 1 && GGFNambu⟦u, 7, 2⟧ == 1 , Print[u, "     ", GGFFeynmanAmplitude⟦u⟧, "\n"]]]
```

374 $\frac{1}{2\hbar^3}\sum_{pqrs}\sum_{tuvw}\sum_{ef} \nabla_{pqrs} \nabla_{tuvw} u^{11}_{ef} \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} dt_3 \int_{-\infty}^{+\infty} dt_4\ G^{(0)}{}_{sp}{}^{11}(t_4,t_4)\ G^{(0)}{}_{wq}{}^{11}(t_3,t_4)\ G^{(0)}{}_{ft}{}^{11}(t_2,t_3)\ G^{(0)}{}_{rd}{}^{11}(t_4,t_1)\ G^{(0)}{}_{ae}{}^{11}(t_5,t_2)\ G^{(0)}{}_{bu}{}^{11}(t_1,t_3)\ G^{(0)}{}_{vc}{}^{11}(t_3,t_5)$

376 $-\frac{1}{2\hbar^3}\sum_{pqrs}\sum_{tuvw}\sum_{ef} \nabla_{pqrs} \nabla_{tuvw} u^{11}_{ef} \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} dt_3 \int_{-\infty}^{+\infty} dt_4\ G^{(0)}{}_{sp}{}^{11}(t_4,t_4)\ G^{(0)}{}_{wq}{}^{11}(t_3,t_4)\ G^{(0)}{}_{ft}{}^{11}(t_2,t_3)\ G^{(0)}{}_{rd}{}^{11}(t_4,t_1)\ G^{(0)}{}_{be}{}^{11}(t_1,t_2)\ G^{(0)}{}_{au}{}^{11}(t_5,t_3)\ G^{(0)}{}_{vc}{}^{11}(t_3,t_5)$

Automated Implementation of Wick's Theorem

for Gorkov's polarization propagator in the ADC scheme

The inequivalent Wick's-theorem contributions are converted into *analytical* expressions.
The latter are in one-to-one correspondence with the amplitudes of the *Feynman graphs* in time representation.

Amplitude of third-order conjoint composite diagrams contributing to Π_{acdb}

with a one-body and two two-body interaction vertices

Conventions

The fully-contracted terms processed henceforth correspond to fully-contracted conjoint composite contributions generated by the application of Wick's theorem to the following matrix element (cf. expansion formula of Gorkov's polarization propagator),

$$\Pi^{g_1g_3g_4g_2}_{acdb}|_{\text{third order}} \equiv +3i\left(\frac{-i}{\hbar}\right)^3 \frac{1}{3!}\int dt_1 \int dt_2 \int dt_3 \langle\Phi_0|T\{\bar{V}(t_1)\bar{V}(t_2)U(t_3)A_{I_a}^{g_1}(t)A_{I_b}^{g_2}(t')A_{I_d}^{\dagger g_4}(t')A_{I_c}^{\dagger g_3}(t)\}|\Phi_0\rangle_{\text{conn}}$$

... ...

```
Framed[19,RoundingRadius->10]" "GGFFeynmanAmplitude[[19]]
Framed[20,RoundingRadius->10]" "GGFFeynmanAmplitude[[20]]
Framed[21,RoundingRadius->10]" "GGFFeynmanAmplitude[[21]]
Framed[22,RoundingRadius->10]" "GGFFeynmanAmplitude[[22]]
Framed[23,RoundingRadius->10]" "GGFFeynmanAmplitude[[23]]
Framed[24,RoundingRadius->10]" "GGFFeynmanAmplitude[[24]]
```

Out[83]:

(1) $\frac{1}{2\hbar^3}\sum_{ef}\sum_{pqrs}\sum_{tuvw} V_{tuvw}\, V_{pqrs}\, u_{ef}^{11}\int_{-\infty}^{+\infty}dt_2\int_{-\infty}^{+\infty}dt_3\int_{-\infty}^{+\infty}dt_4$
$G^{(0)}{}_{pc}^{21}(t_4,t_5)\; G^{(0)}{}_{sq}^{11}(t_4,t_4)\; G^{(0)}{}_{rt}^{11}(t_4,t_3)\; G^{(0)}{}_{wd}^{11}(t_3,t_1)\; G^{(0)}{}_{ue}^{21}(t_3,t_2)\; G^{(0)}{}_{af}^{12}(t_5,t_2)\; G^{(0)}{}_{bv}^{12}(t_1,t_3)$

(2) $\frac{1}{4\hbar^3}\sum_{ef}\sum_{pqrs}\sum_{tuvw} V_{tuvw}\, V_{pqrs}\, u_{ef}^{11}\int_{-\infty}^{+\infty}dt_2\int_{-\infty}^{+\infty}dt_3\int_{-\infty}^{+\infty}dt_4$
$G^{(0)}{}_{pq}^{21}(t_4,t_4)\; G^{(0)}{}_{st}^{11}(t_4,t_3)\; G^{(0)}{}_{re}^{11}(t_4,t_2)\; G^{(0)}{}_{aw}^{12}(t_5,t_3)\; G^{(0)}{}_{bu}^{11}(t_1,t_3)\; G^{(0)}{}_{vd}^{11}(t_3,t_1)\; G^{(0)}{}_{fc}^{11}(t_2,t_5)$

(3) $-\frac{1}{4\hbar^3}\sum_{ef}\sum_{pqrs}\sum_{tuvw} V_{tuvw}\, V_{pqrs}\, u_{ef}^{11}\int_{-\infty}^{+\infty}dt_2\int_{-\infty}^{+\infty}dt_3\int_{-\infty}^{+\infty}dt_4$
$G^{(0)}{}_{pq}^{21}(t_4,t_4)\; G^{(0)}{}_{st}^{11}(t_4,t_3)\; G^{(0)}{}_{re}^{11}(t_4,t_2)\; G^{(0)}{}_{aw}^{12}(t_5,t_3)\; G^{(0)}{}_{bu}^{11}(t_1,t_3)\; G^{(0)}{}_{vc}^{11}(t_3,t_5)\; G^{(0)}{}_{fd}^{11}(t_2,t_1)$

(4) $-\frac{1}{4\hbar^3}\sum_{ef}\sum_{pqrs}\sum_{tuvw} V_{tuvw}\, V_{pqrs}\, u_{ef}^{11}\int_{-\infty}^{+\infty}dt_2\int_{-\infty}^{+\infty}dt_3\int_{-\infty}^{+\infty}dt_4$
$G^{(0)}{}_{pq}^{21}(t_4,t_4)\; G^{(0)}{}_{st}^{11}(t_4,t_3)\; G^{(0)}{}_{re}^{11}(t_4,t_2)\; G^{(0)}{}_{aw}^{12}(t_5,t_3)\; G^{(0)}{}_{ud}^{21}(t_3,t_1)\; G^{(0)}{}_{bv}^{12}(t_1,t_3)\; G^{(0)}{}_{fc}^{11}(t_2,t_5)$

(5) $\frac{1}{\ldots}\sum_{ef}\sum_{pqrs}\sum_{tuvw} V_{tuvw}\, V_{pqrs}\, u_{ef}^{11}\int_{-\infty}^{+\infty}dt_2\int_{-\infty}^{+\infty}dt_3\int_{-\infty}^{+\infty}dt_4$

Entrée [107]:

```
For[u = 1, u <= 1800, u++, If [GGFNambu[[u, 1, 1]] == 1 && GGFNambu[[u, 1, 2]] == 1 && GGFNambu[[u, 2, 1]] == 1
GGFNambu[[u, 3, 1]] == 1 && GGFNambu[[u, 3, 2]] == 1 && GGFNambu[[u, 4, 1]] == 1 && GGFNambu[[u, 4, 2]] == 1 &&
GGFNambu[[u, 7, 1]] == 1 && GGFNambu[[u, 7, 2]] == 1, Print["Framed[",u, ",RoundingRadius->10]\" \"GGFFeynmanAm
```

Output: amplitudes of *conjoint-composite* Feynman diagrams of Gorkov's polarization propagator at third order with $(m,n) = (1,2)$ and Nambu component '*1111*'

Example: fully-contracted second-order Bogoliubov contribution to $\Pi^{1211}_{acdb}(t,t')$ with $(m,n)=(0,2)$

```
CombBog[[1]] = {{1,2},{3,9},{5,6},{7,11},{4,10},{8,12}};
GGFMult[[1]] = 8;
```

- Recasting of the contractions into unperturbed one-body Gorkov-Green's functions (GGFs) and determination of the phase factor:

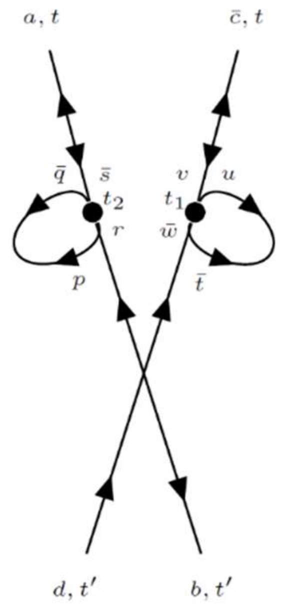
Converting the integers in `CombBog[[1]]` into letters in a string and following the conventions on the second-quantization operators in the matrix elements in the term (P) of the expansion formula, one finds:

$$[pqsatuwdrbvc] \quad \mapsto \quad [pqsrtuwvabdc]$$

$$a^\dagger_p(t_1)a^\dagger_q(t_1)\,a_s(t_1)a_a(t)\,a^\dagger_t(t_2)a^\dagger_u(t_2)\,a_w(t_2)a^\dagger_d(t') \cdot a_r(t_1)a_b(t')\,a_v(t_2)a_{\bar{c}}(t) \mapsto a^\dagger_p(t_1)a^\dagger_q(t_1)a_s(t_1)a_r(t_1)a^\dagger_t(t_2)a^\dagger_u(t_2)a_w(t_2)a_v(t_2) \cdot a_a(t)a_b(t')a^\dagger_d(t')a_{\bar{c}}(t)\ ,$$

the number of transpositions necessary to restore the canonical sequence of 2nd-quantization operators is *12*. Two additional *sign-changing* operations are performed, in order to obey the conventions for the conversion of contractions into one-body GGFs: factor $(-1)^2$

$$(-1)^T(-i)^{n+m+1}i^{2n+m+2} \quad \mapsto \quad i^3(-1)^{14} = -i$$



- Introducing the rest of the necessary symbols and the multiplicity factor in `GGFMult[[1]]` the Feynman amplitude is finally found:

$$-\frac{i}{4\hbar^2}\sum_{pqrs}\sum_{tuvw}\bar{v}_{\bar{p}qr\bar{s}}\bar{v}_{t\bar{u}v\bar{w}}\int_{-\infty}^{+\infty}\mathrm{d}t_1\int_{-\infty}^{+\infty}\mathrm{d}t_2\; G^{(0)\ 21}_{pq}(t_1,t_1^+)G^{(0)\ 12}_{as}(t,t_1) \cdot G^{(0)\ 21}_{tu}(t_2,t_2^+)G^{(0)\ 22}_{\bar{d}w}(t',t_2)G^{(0)\ 12}_{r\bar{b}}(t_1,t')G^{(0)\ 12}_{vc}(t_1,t)$$

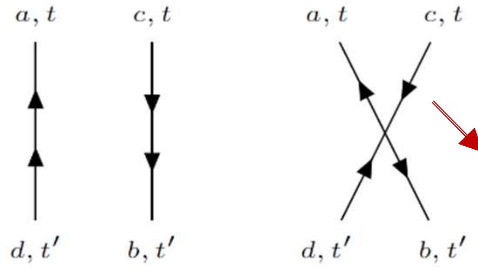
Leading and Subleading Order Diagrams

of the polarization propagator

Example: perturbation expansion of $\Pi^{1111}_{acdb}(t,t')$

- Zeroth order:

$$\Pi^{1111\,(0a)}_{acdb}(t,t') = -i\, G^{(0)\,11}_{ad}(t,t'^{+})\, G^{(0)\,11}_{cb}(t^{+},t')$$



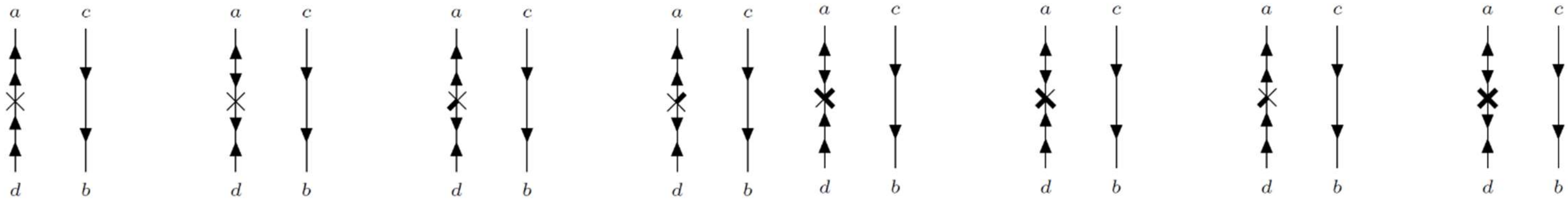
(0a) (0b)

$$\Pi^{1111\,(0b)}_{acdb}(t,t') = i\, G^{(0)\,12}_{a\bar{b}}(t,t')\, G^{(0)\,21}_{\bar{d}c}(t'^{+},t^{+})$$

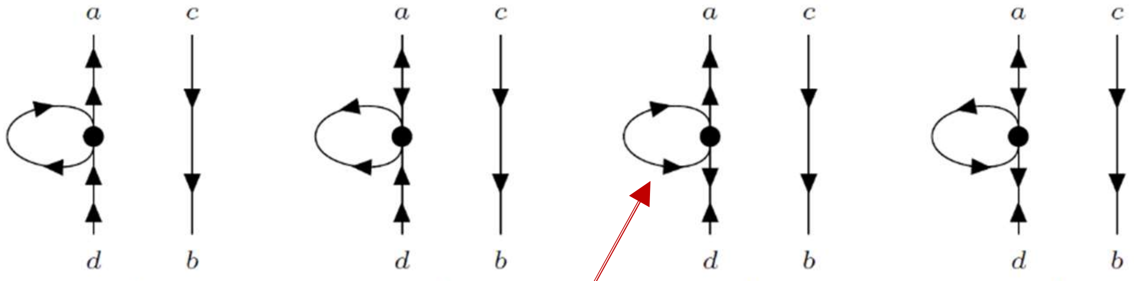
i.e. the LO disjoint direct and Bogoliubov diagrams!

- First order:

left-dressed direct diagrams



(1′aa) (1′ab) (1′ac) (1′ad) (1′ae) (1′af) (1′ag) (1′ah)



(1′ai) (1′aj) (1′ak) (1′al)

Anomalous loop!

$$\Pi^{1111\,(1'ak)}_{acdb}(t,t') = +\frac{1}{2\hbar}\sum_{pqrs} \bar{v}_{p\bar{q}\bar{r}s} \int_{-\infty}^{+\infty} \mathrm{d}t_1 G^{(0)\,11}_{ap}(t,t_1) G^{(0)\,12}_{sr}(t_1,t_1^{+}) G^{(0)\,11}_{qd}(t_1,t'^{+}) G^{(0)\,22}_{bc}(t',t^{+})$$

Ellipsis: *right-dressed* direct diagrams

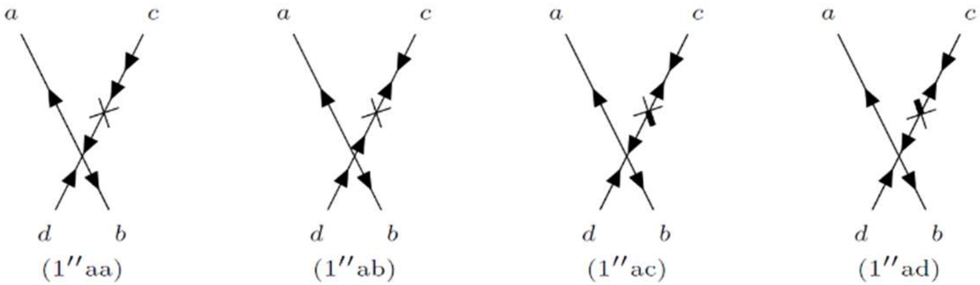
SUBLEADING ORDER DIAGRAMS

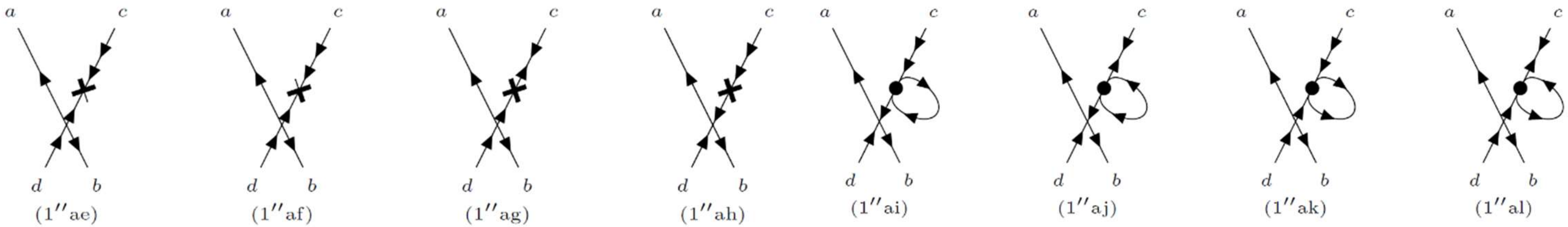
of the polarization propagator

Example: perturbation expansion of $\Pi^{1111}_{acdb}(t, t')$

- First order:

  - *antidiagonally-dressed* Bogoliubov diagrams

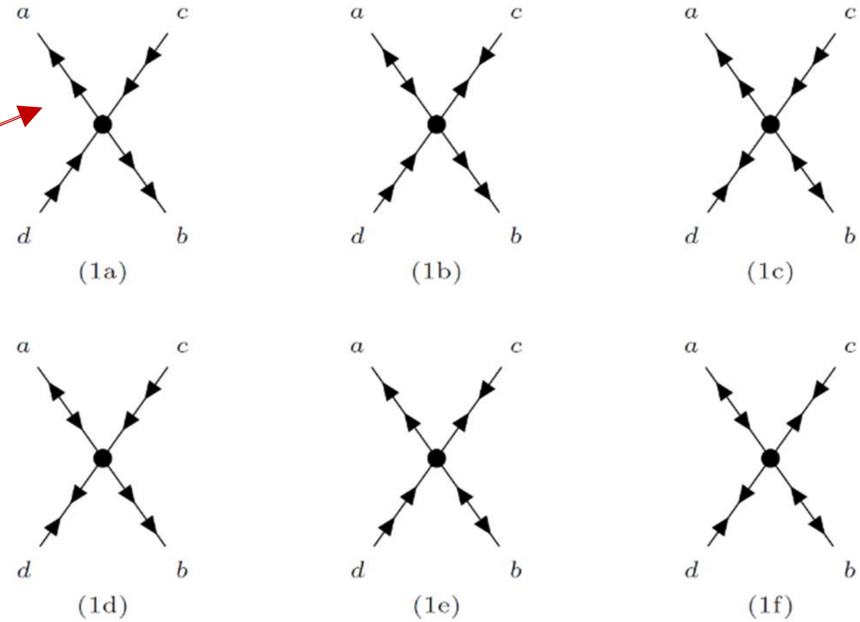




  - conjoint skeleton diagrams

$$\Pi^{1111\,(1a)}_{acdb}(t, t') = -\frac{1}{\hbar} \sum_{pqrs} \bar{v}_{pqrs} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \, G^{(0)\,11}_{bq}(t', t_1)$$

$$G^{(0)\,11}_{ap}(t, t_1) G^{(0)\,11}_{rc}(t_1, t^+) \, G^{(0)\,11}_{sd}(t_1, t'^+)$$



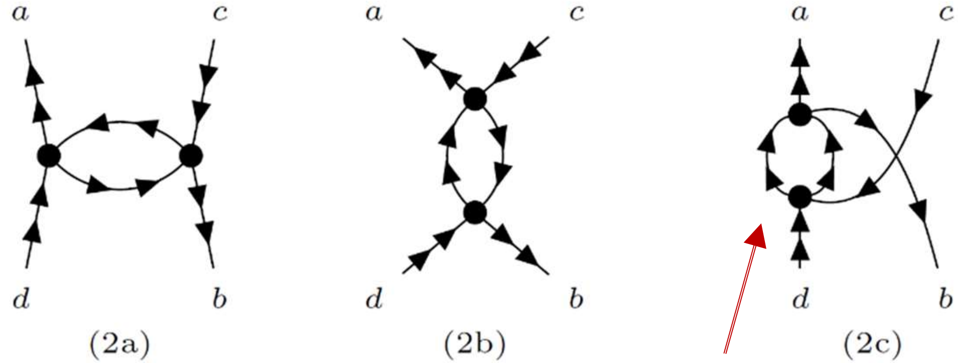
Ellipsis: *diagonally-dressed* Bogoliubov diagrams

Higher Order diagrams

of the polarization propagator

Example: perturbation expansion of $\Pi^{1111}_{acdb}(t,t')$

- Second order:
  - conjoint *skeleton* diagrams



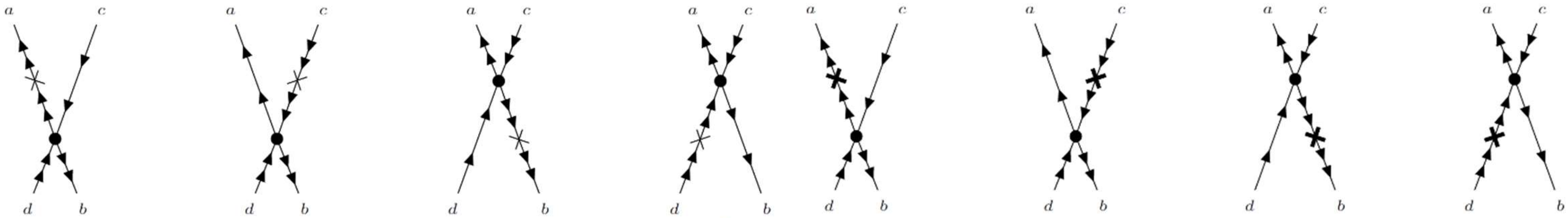
(2a) (2b) (2c)

Ellipsis: conjoint skeleton diagrams containing anomalous propagators

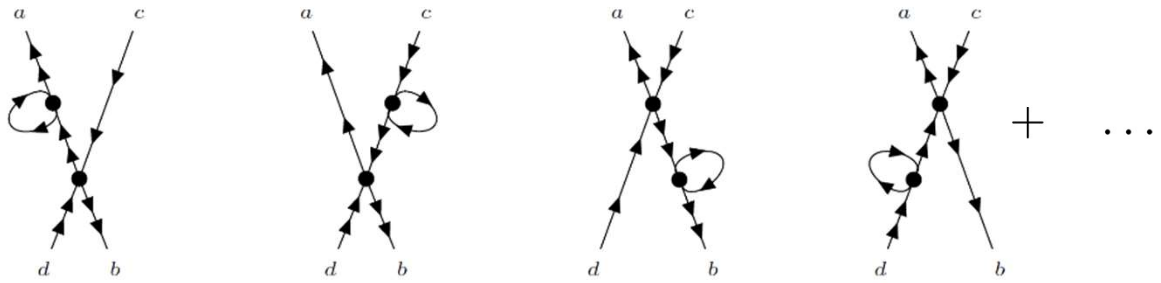
Equivalent propagators!

$$\Pi^{1111\,(2c)}_{acdb}(t,t') = -\frac{i}{2\hbar^2}\sum_{\substack{pqrs\\tuvw}} \bar{v}_{pqrs}\bar{v}_{tuvw}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\, G^{(0)\,11}_{wp}(t_2,t_1)\, G^{(0)\,11}_{sd}(t_1,t'^{+})\, G^{(0)\,11}_{at}(t,t_2)\, G^{(0)\,11}_{vq}(t_2,t_1)\, G^{(0)\,11}_{bu}(t',t_1)\, G^{(0)\,11}_{rc}(t_1,t^{+})$$

  - conjoint *composite* diagrams



(2′a) (2′b) (2′c) (2′d) (2′e) (2′f) (2′g) (2′h)



(2′i) (2′j) (2′k) (2′l)

Ellipsis: conjoint composite diagrams containing anomalous propagators

Other contributions: disjoint direct and Bogoliubov diagrams

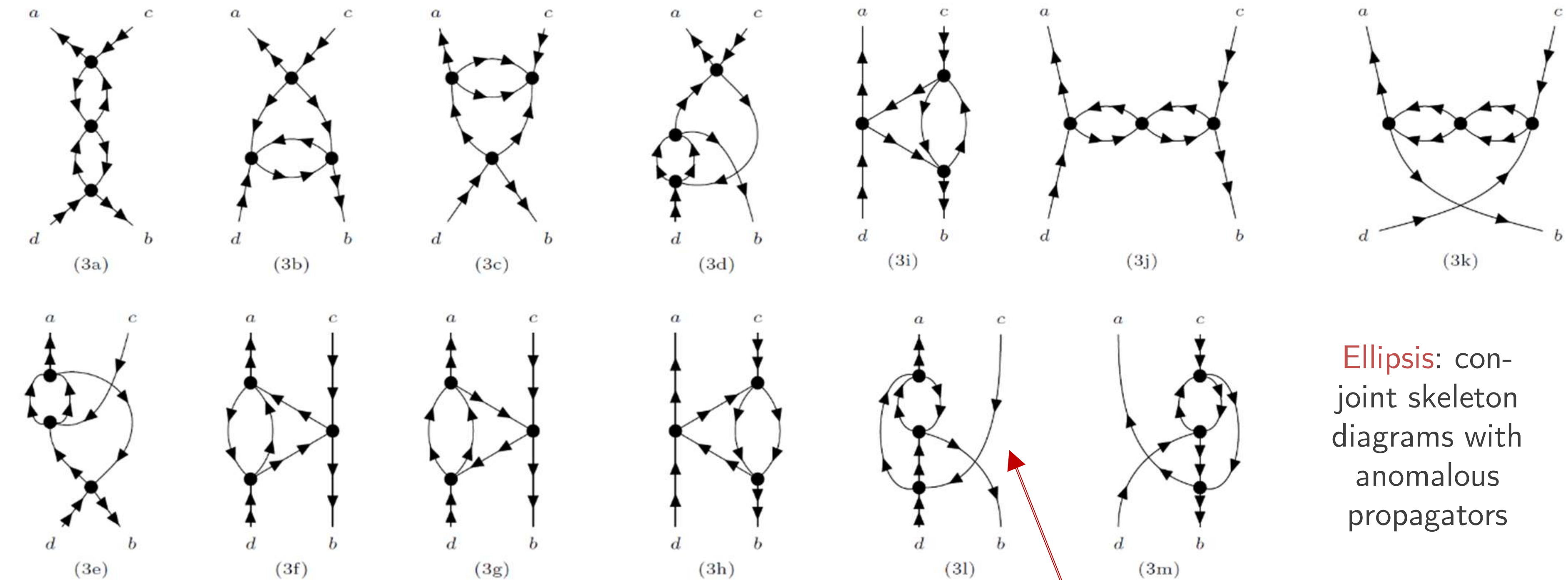
HIGHER ORDER DIAGRAMS

of the polarization propagator

Example: perturbation expansion of $\Pi^{1111}_{acdb}(t,t')$

- Third order*:*

  - conjoint *skeleton* diagrams



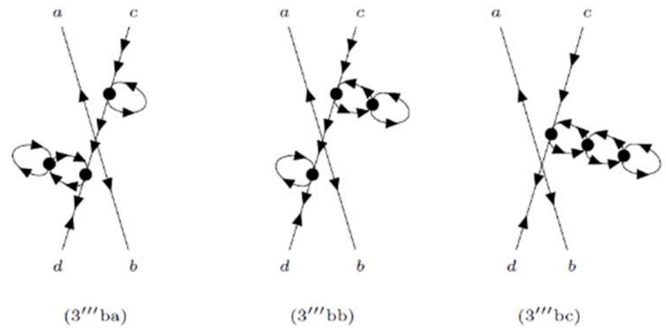
(3a) (3b) (3c) (3d) (3i) (3j) (3k)

(3e) (3f) (3g) (3h) (3l) (3m)

Ellipsis: conjoint skeleton diagrams with anomalous propagators

$$\Pi^{1111\,(3l)}_{acdb}(t,t') = -\frac{1}{\hbar^3}\sum_{pqrs}\sum_{tuvw}\sum_{klmn}\bar{v}_{pqrs}\bar{v}_{tuvw}\bar{v}_{klmn}\int_{-\infty}^{+\infty}\mathrm{d}t_1\int_{-\infty}^{+\infty}\mathrm{d}t_2\int_{-\infty}^{+\infty}\mathrm{d}t_3$$
$$\cdot G^{(0)\,11}_{wp}(t_2,t_1)\,G^{(0)\,11}_{st}(t_1,t_2):\;G^{(0)\,11}_{rk}(t_1,t_3)\,G^{(0)\,11}_{aq}(t,t_1)G^{(0)\,11}_{bu}(t',t_2)$$
$$\cdot G^{(0)\,11}_{nd}(t_3,t'^{+})\,G^{(0)\,11}_{vl}(t_2,t_3)\,G^{(0)\,11}_{mc}(t_3,t^{+})$$

  - *antidiagonally-dressed* disjoint Bogoliubov diagrams



(3‴ba) (3‴bb) (3‴bc) (3‴bd)

$+\;\cdots$

Ellipsis: antidiagonally-dr.disjoint Bogoliubov diagrams containing *one-body vertices* and more than one anomalous propagator

HIGHER ORDER DIAGRAMS

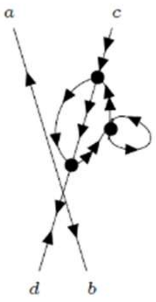
of the polarization propagator

Example: perturbation expansion of $\Pi^{1111}_{acdb}(t,t')$

▶ Third order:

antidiagonally-dressed disjoint Bogoliubov diagrams

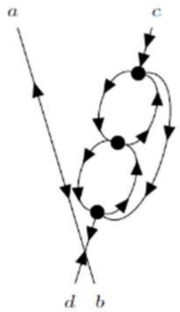
(3'''be)



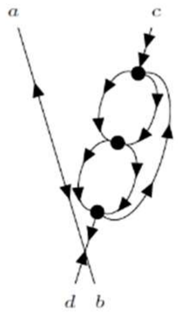
(3'''bf)

(3'''bg)

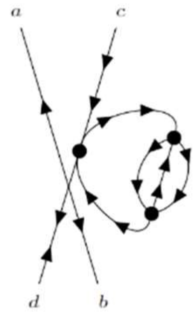
(3'''bh)



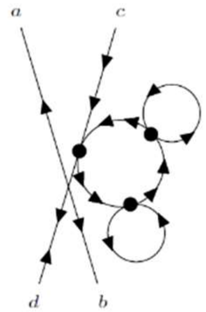
(3'''bi)



(3'''bj)



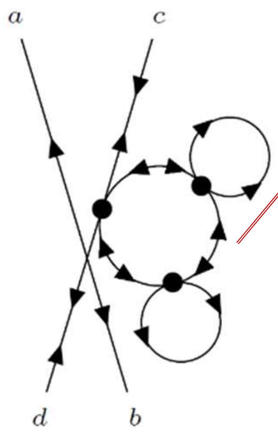
(3'''bk)



(3'''bl)

$+ \quad \dots$

$$\Pi^{1111\,(3'''bm)}_{acdb}(t,t') = -\frac{1}{8\hbar^3}\sum_{pqrs}\sum_{tuvw}\sum_{klmn} \bar{v}_{\bar{p}q\bar{r}s}\bar{v}_{\bar{t}u\bar{v}u}\bar{v}_{\bar{k}l\bar{m}n}\int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{+\infty}dt_2 \int_{-\infty}^{+\infty}dt_3\, G^{21\,(0)}_{tu}(t_2,t_2^+)G^{21\,(0)}_{pq}(t_1,t_1^+)G^{12\,(0)}_{wm}(t_2,t_3)G^{12\,(0)}_{sv}(t_1,t_2)\, G^{12\,(0)}_{nr}(t_3,t_1)G^{21\,(0)}_{kc}(t_3,t)G^{21\,(0)}_{\bar{d}l}(t_3,t')G^{12\,(0)}_{a\bar{b}}(t,t')$$

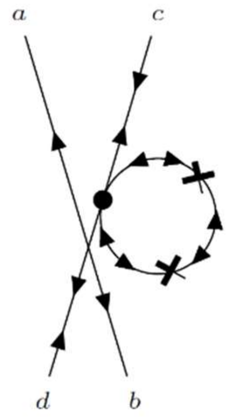


(3'''bm)

Equivalent vertices

& two anomalous loops!

Remark: *Equivalent vertices* can be also of one-body type



(3'''bn)

Ellipsis: antidiagonally-dr. disjoint Bogoliubov diagrams containing *one-body vertices* and more than one anomalous propagator

Other contributions: conjoint composite, disjoint direct and further disjoint Bogoliubov diagrams

Algebraic Diagrammatic Construction

for the polarization propagator

Starting-point: the one-body transition operator $\mathcal{D} = \sum_{rs} D_{rs} a_r^\dagger a_s$

J. Schirmer, *Phys. Rev. A* **26**, 5, 2395-2416 (1982)

for particle-number conserving operators, such as EM trans. oper. $D_{rs} \equiv D_{rs}^{11}$ or D_{rs}^{22}

Thanks to the complex-conj. property, one may consider only $\Pi^{+\,g_1 g_3 g_4 g_2}_{acdb}(\omega)$

Defining the transition function as

$$T(\omega) \equiv \sum_{abcd} D^*_{ac} \Pi^{+\,1111}_{acdb}(\omega) D_{db}$$

Lehmann's repr. permits to write

$$T(\omega) \equiv \mathbf{T}^\dagger (\omega \mathbb{1} - \mathbf{\Delta}) \mathbf{T}$$

where $\Delta_{jk} \equiv \langle \Psi_j | \Omega - \Omega_0 | \Psi_k \rangle / \hbar \implies$ *secular matrix* and $T_k = \langle \Psi_k | \mathcal{D} | \Psi_0 \rangle \implies$ *vector of transition ampl.*

▶ Construction of the ADC ansatz, similar to the one for the self energy ($\mathbf{\Sigma}^{(\mathrm{dyn})\,+}_{ab}$)

$$T(\omega) \equiv \mathbf{F}^\dagger (\omega \mathbb{1} - \mathbf{K} - \mathbf{C})^{-1} \mathbf{F}$$

where $\mathbf{K} \Rightarrow$ *matrix of diff. betw. the eigenvalues associated with* Ω_U: $K_{ij,kl} = \delta_{ik}\delta_{jl}(\omega_i - \omega_j)/\hbar$

As for the self-energy, the matrices admit an order by order expansion

$$\mathbf{C} \equiv \mathbf{C}^{(1)} + \mathbf{C}^{(2)} + \ldots \qquad \mathbf{F} \equiv \mathbf{F}^{(0)} + \mathbf{F}^{(1)} + \mathbf{F}^{(2)} + \ldots$$

Again the geometric series gives...

$$T(\omega) \overset{\text{ADC}}{=} \mathbf{F}^\dagger (\omega \mathbb{1} - \mathbf{K})^{-1} \sum_{n=0}^{+\infty} \left\{ \mathbf{C} (\omega \mathbb{1} - \mathbf{K})^{-1} \right\}^n \mathbf{F}$$

Matching procedure with the standard pert. expansion yields the expressions for $\mathbf{F}$, $\mathbf{C}$ and $\mathbf{K}$

$$T(\omega) \equiv T(\omega)^{(0)} + T(\omega)^{(1)} + T(\omega)^{(2)} + \ldots$$

The ADC splits the problem of determing $\mathbf{T}$ into two tasks: the *construction* of the modified transition ampl. $\mathbf{F}$ and the *diagonalization* proc. for the modified. interaction matrix, $\mathbf{C} + \mathbf{K}$.

Goldstone Diagrams

for the polarization propagator

In the ADC for the polarization propag. in energy repres. time integrations are carried out by considering the $m{+}n{+}2!$ possible orderings of the time indices in the $n{+}m$ vertices at order l

- The time-ordered Feynman (≡ *Goldstone*) amplitudes are Fourier-transformed. The $m{+}n$ time integrations are performed, exploiting the Fourier representation of the *Dirac deltas* and the *theta functions*. The ensuing expressions are given in terms of *spectroscopic amplitudes*.

Example: code for the amplitudes of *disjoint-direct* Goldstone diagrams contributing to $\Pi^{+\,2221}_{acdb}(t,t')$ at third order with $(m,n) = (2,1)$

Entrée [191]:

```
For[r = 1, r <= NcMax, r++,
 For[j = 1, j <= 6, j++,
  For[i = 1, i <= 2, i++,
    If[GGFSParticle[[r, j, i]] == 1 && GGFNambu[[r, j, i]] == 1, SPLIndex[[1]] = "p"];
    If[GGFSParticle[[r, j, i]] == 1 && GGFNambu[[r, j, i]] == 2, SPLIndex[[1]] = OverBar["p"]];
    If[GGFSParticle[[r, j, i]] == 2 && GGFNambu[[r, j, i]] == 1, SPLIndex[[2]] = "q"];
    If[GGFSParticle[[r, j, i]] == 2 && GGFNambu[[r, j, i]] == 2, SPLIndex[[2]] = OverBar["q"]];
    If[GGFSParticle[[r, j, i]] == 3 && GGFNambu[[r, j, i]] == 1, SPLIndex[[3]] = "s"];
    If[GGFSParticle[[r, j, i]] == 3 && GGFNambu[[r, j, i]] == 2, SPLIndex[[3]] = OverBar["s"]];
    If[GGFSParticle[[r, j, i]] == 4 && GGFNambu[[r, j, i]] == 1, SPLIndex[[4]] = "r"];
    If[GGFSParticle[[r, j, i]] == 4 && GGFNambu[[r, j, i]] == 2, SPLIndex[[4]] = OverBar["r"]];
    If[GGFSParticle[[r, j, i]] == 5 && GGFNambu[[r, j, i]] == 1, SPLIndex[[5]] = "e"];
    If[GGFSParticle[[r, j, i]] == 5 && GGFNambu[[r, j, i]] == 2, SPLIndex[[5]] = OverBar["e"]];
    If[GGFSParticle[[r, j, i]] == 6 && GGFNambu[[r, j, i]] == 1, SPLIndex[[6]] = "f"];
    If[GGFSParticle[[r, j, i]] == 6 && GGFNambu[[r, j, i]] == 2, SPLIndex[[6]] = OverBar["f"]];
    If[GGFSParticle[[r, j, i]] == 7 && GGFNambu[[r, j, i]] == 1, SPLIndex[[7]] = "g"];
    If[GGFSParticle[[r, j, i]] == 7 && GGFNambu[[r, j, i]] == 2, SPLIndex[[7]] = OverBar["g"]];
    If[GGFSParticle[[r, j, i]] == 8 && GGFNambu[[r, j, i]] == 1, SPLIndex[[8]] = "h"];
    If[GGFSParticle[[r, j, i]] == 8 && GGFNambu[[r, j, i]] == 2, SPLIndex[[8]] = OverBar["h"]];
    ];
  ];
 Pref = StringForm["``", (-1)^AmplSign[[r]] 3 (-I)^4/ℏ^3 (1/-I)^4 GGFMult[[r]]/96 UMult[[NuIdx]]];
 Vind1 = StringForm["`1``2``3``4`",SPLIndex[[1]], SPLIndex[[2]], SPLIndex[[4]], SPLIndex[[3]]];
 Uins1 = Superscript[Superscript[Subscript["u",Subscript[•,SPLIndex[[5]] SPLIndex[[6]]]],UNambu[[NuIdx,1]]],UNa
 Uins2 = Superscript[Superscript[Subscript["u",Subscript[•,SPLIndex[[7]] SPLIndex[[8]]]],UNambu[[NuIdx,1]]],UNa
 Symb1 = Subscript["∑",Subscript[•,Subscript[k,1]Subscript[k,2]Subscript[k,3]]]
   Subscript["∑",Subscript[•,Subscript[k,4]Subscript[k,5]Subscript[k,6]]]
   Subscript["∑",Subscript[•,pqrs]] Subscript["∑",Subscript[•,efgh]];
 Symb2 = Subscript[OverBar["v"],Subscript[•,Vind1]] Uins1 Uins2;
  TimeIntegralSolutions1 = Solve[Coefficient[GGFTotalPolynomial[[r]], Subscript[t,TimeClassOrdPlus[[T, 2]]]] ==
     Coefficient[GGFTotalPolynomial[[r]], Subscript[t,TimeClassOrdPlus[[T, 3]]]] == 0 &&
     Coefficient[GGFTotalPolynomial[[r]], Subscript[t,TimeClassOrdPlus[[T, 4]]]] == 0, {ϵ, γ, ι}];
  {ϵ, γ, ι} = TimeIntegralSolutions1 // Values // Flatten;
  TimeIntegralSolutions2 = Solve[Coefficient[GGFTotalPolynomial[[r]], Subscript[t,TimeClassOrdPlus[[T, 1]]]] ==
  {ƒ} = TimeIntegralSolutions2 // Values // Flatten;
  GGFGoldstoneAmplitude[[r]] = StringForm["`1``2``3`·`4`·`5`·`6`·`7`", Pref, Symb1,Symb2,
      GGFAmplitude[[r, 1]] GGFAmplitude[[r, 2]]/(ϵ + I Superscript[η,"(0)"]), GGFAmplitude[[r, 3]] GGFAmplitude[
      GGFAmplitude[[r, 5]]/(γ + I Superscript[η,"(2)"]), GGFAmplitude[[r, 6]]/(ι + I Superscript[η,"(3)"])];
  Clear[ƒ];
  Clear[ϵ];
  Clear[γ];
  Clear[ι];
  Clear[TimeIntegralSolutions1];
  Clear[TimeIntegralSolutions2];
  ];
```

More examples of Goldstone diagrams & ADC scheme ⇝ *Appendix*

Output: amplitudes of *disjoint-direct* Goldstone diagrams of $\Pi^{+\,2221}_{acdb}(t,t')$ at third order with $(m,n) = (2,1)$ and time ordering $t > t_1 > t_2 > t_3 > t'$

Non-identical one-body vertices of type u^{21}_{ef} and u^{22}_{ef}: multipl. = *1*

Amplitude of third-order left-and-right-dressed disjoint direct diagrams contributing to Π_{acdb}

with one two-body and two one-body interaction vertices

Conventions

The fully-contracted terms processed henceforth correspond to fully-contracted left-and-right-dressed disjoint direct contributions generated by the application of Wick's theorem to the following matrix element (cf. expansion formula of Gorkov's polarization propagator),

$$\Pi_{acdb}{}^{g_1 g_3 g_4 g_2}\big|_{\text{second order}} = -i\left(\frac{-i}{\hbar}\right)^3 \frac{1}{3!}\, 3 \int dt_1 \int dt_2 \int dt_3\, \langle \Phi_0 |\, T\{ \bar{V}(t_1)\, U(t_2)\, U(t_3)\, A_{la}{}^{g_1}(t) A_{lb}{}^{g_2}(t') A_{ld}{}^{\dagger g_4}(t') A_{lc}{}^{\dagger g_3}(t) | \Phi_0 \rangle$$

where the one-body and the two-body potential insertions, $\bar{V}(t_1)\, U(t_2)$ and $U(t_3)$, take the form

$$\bar{V}(t_1) = \frac{1}{4} \sum_{pqrs} \bar{V}_{pqrs}\, a_p^\dagger(t_1)\, a_q^\dagger(t_1)\, a_s(t_1)\, a_r(t_1),$$

$$U(t_2) = \frac{1}{2}\sum_{ef} \left[u_{ef}{}^{11} a_e{}^{\dagger}(t_2)\, a_f(t_2) + u_{ef}{}^{22} a_{\bar e}(t_2)\, a_{\bar f}{}^{\dagger}(t_2) + u_{ef}{}^{12} a_e{}^{\dagger}(t_2)\, a_{\bar f}{}^{\dagger}(t_2) + u_{ef}{}^{21} a_{\bar e}(t_2)\, a_f(t_2) \right],$$

$$U(t_3) = \frac{1}{2}\sum_{gh} \left[u_{gh}{}^{11} a_g{}^{\dagger}(t_3)\, a_h(t_3) + u_{gh}{}^{22} a_{\bar g}(t_3)\, a_{\bar h}{}^{\dagger}(t_3) + u_{gh}{}^{12} a_g{}^{\dagger}(t_3)\, a_{\bar h}{}^{\dagger}(t_3) + u_{gh}{}^{21} a_{\bar g}(t_3)\, a_h(t_3) \right].$$

... ...

Results with T = 4, corresponding to {1,2,3,4,5}

```
In[152]:= For[u = 1, u ≤ 24, u++, Print[u, "     ", GGFGoldstoneAmplitude[[u]]]]
```

1 $-\frac{1}{8\hbar^3}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\sum_{pqrs}\sum_{efgh}\bar{V}_{\bar p q \bar r s}\, u^{21}_{\bar e f}\, u^{22}_{g \bar h}\, \frac{\chi_p^{2k_1}\chi_s^{1k_2}\bar\chi_e^{2k_2}\bar\chi_q^{1k_1}}{-\frac{\Omega_{k_2}-\Omega_{k_3}}{\hbar}+i\eta}\cdot\frac{\chi_a^{2k_3}\chi_g^{1k_4}\bar\chi_b^{2k_4}\bar\chi_r^{2k_3}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_4}-\Omega_{k_5}}{\hbar}+i\eta'}\cdot\frac{\chi_f^{1k_5}\bar\chi_d^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_5}+\Omega_{k_6}}{\hbar}+i\eta''}\cdot\frac{-\Upsilon_h^{2k_6}\bar\Upsilon_c^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_6}}{\hbar}+i\eta'''}$

2 $-\frac{1}{8\hbar^3}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\sum_{pqrs}\sum_{efgh}\bar{V}_{\bar p q \bar r s}\, u^{21}_{\bar e f}\, u^{22}_{g \bar h}\, \frac{\chi_p^{2k_1}\chi_s^{1k_2}\bar\chi_e^{2k_2}\bar\chi_q^{1k_1}}{-\frac{\Omega_{k_2}-\Omega_{k_3}}{\hbar}+i\eta}\cdot\frac{-\Upsilon_g^{1k_4}\chi_a^{2k_3}\bar\Upsilon_c^{2k_4}\bar\chi_r^{2k_3}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_5}-\Omega_{k_6}}{\hbar}+i\eta'}\cdot\frac{\chi_f^{1k_5}\bar\chi_d^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_4}+\Omega_{k_5}}{\hbar}+i\eta''}\cdot\frac{-\Upsilon_b^{1k_6}\bar\Upsilon_h^{1k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_4}}{\hbar}+i\eta'''}$

3 $\frac{1}{8\hbar^3}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\sum_{pqrs}\sum_{efgh}\bar{V}_{\bar p q \bar r s}\, u^{21}_{\bar e f}\, u^{22}_{\bar g h}\, \frac{\chi_p^{2k_1}\chi_s^{1k_2}\bar\chi_e^{2k_2}\bar\chi_q^{1k_1}}{-\frac{-2\Omega_0+\Omega_{k_2}+\Omega_{k_3}}{\hbar}+i\eta}\cdot\frac{-\Upsilon_b^{1k_3}\chi_a^{2k_4}\bar\Upsilon_r^{2k_3}\bar\chi_g^{2k_4}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_3}-\Omega_{k_5}}{\hbar}+i\eta'}\cdot\frac{\chi_h^{2k_5}\bar\chi_d^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_4}}{\hbar}+i\eta''}\cdot\frac{-\Upsilon_f^{1k_6}\bar\Upsilon_c^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_4}+\Omega_{k_6}}{\hbar}+i\eta'''}$

4 $-\frac{1}{8\hbar^3}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\sum_{pqrs}\sum_{efgh}\bar{V}_{\bar p q \bar r s}\, u^{21}_{\bar e f}\, u^{22}_{g \bar h}\, \frac{\chi_p^{2k_1}\chi_s^{1k_2}\bar\chi_e^{2k_2}\bar\chi_q^{1k_1}}{-\frac{-2\Omega_0+\Omega_{k_2}+\Omega_{k_3}}{\hbar}+i\eta}\cdot\frac{-\Upsilon_b^{1k_3}\chi_g^{1k_4}\bar\Upsilon_r^{2k_3}\bar\chi_d^{2k_4}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_3}-\Omega_{k_4}}{\hbar}+i\eta'}\cdot\frac{\chi_a^{2k_5}\bar\chi_h^{1k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_5}}{\hbar}+i\eta''}\cdot\frac{-\Upsilon_f^{1k_6}\bar\Upsilon_c^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_5}+\Omega_{k_6}}{\hbar}+i\eta'''}$

5 $\frac{1}{8\hbar^3}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\sum_{pqrs}\sum_{efgh}\bar{V}_{\bar p q r s}\, u^{21}_{\bar e f}\, u^{22}_{\bar g h}\, \frac{\chi_p^{2k_1}\chi_s^{1k_2}\bar\chi_e^{2k_2}\bar\chi_q^{1k_1}}{-\frac{-2\Omega_0+\Omega_{k_2}+\Omega_{k_3}}{\hbar}+i\eta}\cdot\frac{\chi_g^{1k_4}\chi_r^{1k_3}\bar\chi_b^{2k_4}\bar\chi_d^{2k_3}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_3}-\Omega_{k_4}}{\hbar}+i\eta'}\cdot\frac{\chi_a^{2k_5}\bar\chi_f^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_6}}{\hbar}+i\eta''}\cdot\frac{-\Upsilon_h^{2k_6}\bar\Upsilon_c^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_5}+\Omega_{k_6}}{\hbar}+i\eta'''}$

6 $\frac{1}{8\hbar^3}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\sum_{pqrs}\sum_{efgh}\bar{V}_{\bar p q r s}\, u^{21}_{\bar e \bar f}\, u^{22}_{\bar g h}\, \frac{\chi_p^{2k_1}\chi_s^{1k_2}\bar\chi_e^{2k_2}\bar\chi_q^{1k_1}}{-\frac{-2\Omega_0+\Omega_{k_2}+\Omega_{k_3}}{\hbar}+i\eta}\cdot\frac{-\Upsilon_g^{1k_4}\chi_r^{1k_3}\bar\Upsilon_c^{2k_4}\bar\chi_d^{2k_3}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_3}-\Omega_{k_6}}{\hbar}+i\eta'}\cdot\frac{\chi_a^{2k_5}\bar\chi_f^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_4}}{\hbar}+i\eta''}\cdot\frac{-\Upsilon_b^{1k_6}\bar\Upsilon_h^{1k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_4}+\Omega_{k_5}}{\hbar}+i\eta'''}$

7 $-\frac{1}{8\hbar^3}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\sum_{pqrs}\sum_{efgh}\bar{V}_{\bar p q r s}\, u^{21}_{\bar e f}\, u^{22}_{g \bar h}\, \frac{\chi_p^{2k_1}\chi_s^{1k_2}\bar\chi_e^{2k_2}\bar\chi_q^{1k_1}}{-\frac{\Omega_{k_2}-\Omega_{k_3}}{\hbar}+i\eta}\cdot\frac{-\Upsilon_r^{1k_3}\chi_a^{2k_4}\bar\Upsilon_c^{2k_3}\bar\chi_g^{2k_4}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_5}-\Omega_{k_6}}{\hbar}+i\eta'}\cdot\frac{\chi_h^{2k_5}\bar\chi_d^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_4}+\Omega_{k_6}}{\hbar}+i\eta''}\cdot\frac{-\Upsilon_b^{1k_6}\bar\Upsilon_f^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_4}}{\hbar}+i\eta'''}$

Output: amplitudes of *disjoint-direct* Goldstone diagrams of $\Pi^{+\,2221}_{acdb}(t,t')$ at third order with $(m,n) = (2,1)$ and time ordering $t > t_1 > t_2 > t_3 > t'$

Identical one-body vertices of type u^{11}_{ef}: multipl. = *2!*

Amplitude of third-order left-and-right-dressed disjoint direct diagrams contributing to Π_{acdb}

with a two-body and two one-body interaction vertices

Conventions

The fully-contracted terms processed henceforth correspond to fully-contracted left-and-right-dressed disjoint direct contributions generated by the application of Wick's theorem to the following matrix element (cf. expansion formula of Gorkov's polarization propagator),

$$\Pi^{g_1g_3g_4g_2}_{acdb}|_{\text{third order}} \equiv -3i\left(\frac{-i}{\hbar}\right)^3\frac{1}{3!}\int dt_1\int dt_2\int dt_3\,\langle\Phi_0|T\{\bar{V}(t_1)U(t_2)U(t_3)A_{I_a}^{g_1}(t)A_{I_b}^{g_2}(t')A_{I_d}^{\dagger g_4}(t')A_{I_c}^{\dagger g_3}(t)\}|\Phi_0\rangle_{\text{conn}}$$

. . .   .

Results with T = 4, corresponding to {1,2,3,4,5}

Entrée [ ]:

```
For[u = 1, u <= NcMax, u++, Print["Framed[",u,",RoundingRadius->10]\" \"GGFGoldstoneAmplitude[[",u,"]]"]]
```

Entrée [226]:

```
Framed[1,RoundingRadius->10]" "GGFGoldstoneAmplitude[[1]]
Framed[2,RoundingRadius->10]" "GGFGoldstoneAmplitude[[2]]
Framed[3,RoundingRadius->10]" "GGFGoldstoneAmplitude[[3]]
Framed[4,RoundingRadius->10]" "GGFGoldstoneAmplitude[[4]]
Framed[5,RoundingRadius->10]" "GGFGoldstoneAmplitude[[5]]
Framed[6,RoundingRadius->10]" "GGFGoldstoneAmplitude[[6]]
Framed[7,RoundingRadius->10]" "GGFGoldstoneAmplitude[[7]]
Framed[8,RoundingRadius->10]" "GGFGoldstoneAmplitude[[8]]
Framed[9,RoundingRadius->10]" "GGFGoldstoneAmplitude[[9]]
Framed[10,RoundingRadius->10]" "GGFGoldstoneAmplitude[[10]]
Framed[11,RoundingRadius->10]" "GGFGoldstoneAmplitude[[11]]
Framed[12,RoundingRadius->10]" "GGFGoldstoneAmplitude[[12]]
Framed[13,RoundingRadius->10]" "GGFGoldstoneAmplitude[[13]]
Framed[14,RoundingRadius->10]" "GGFGoldstoneAmplitude[[14]]
Framed[15,RoundingRadius->10]" "GGFGoldstoneAmplitude[[15]]
Framed[16,RoundingRadius->10]" "GGFGoldstoneAmplitude[[16]]
Framed[17,RoundingRadius->10]" "GGFGoldstoneAmplitude[[17]]
Framed[18,RoundingRadius->10]" "GGFGoldstoneAmplitude[[18]]
```

Out[226]:

(1) $\frac{1}{8\hbar^3}\sum_{efgh}\sum_{pqrs}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\bar{V}_{pqrs}u^{11}_{ef}u^{11}_{gh}\cdot -\frac{\chi_p^{2k_1}\Upsilon_s^{1k_2}\bar{\chi}_q^{1k_1}\bar{\Upsilon}_e^{1k_2}}{-\frac{2\Omega_0-\Omega_{k_2}-\Omega_{k_3}}{\hbar}+i\eta^{(0)}}\cdot -\frac{\chi_a^{2k_3}\Upsilon_b^{1k_4}\bar{\chi}_r^{2k_3}\bar{\Upsilon}_g^{1k_4}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_4}-\Omega_{k_5}}{\hbar}+i\eta^{(1)}}\cdot\frac{\chi_f^{1k_5}\bar{\chi}_d^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_5}+\Omega_{k_6}}{\hbar}+i\eta^{(2)}}\cdot -\frac{\Upsilon_h^{1k_6}\bar{\Upsilon}_c^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_6}}{\hbar}+i\eta^{(3)}}$

(2) $-\frac{1}{8\hbar^3}\sum_{efgh}\sum_{pqrs}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\bar{V}_{pqrs}u^{11}_{ef}u^{11}_{\bar{g}h}\cdot -\frac{\chi_p^{2k_1}\Upsilon_s^{1k_2}\bar{\chi}_q^{1k_1}\bar{\Upsilon}_e^{1k_2}}{-\frac{2\Omega_0-\Omega_{k_2}-\Omega_{k_3}}{\hbar}+i\eta^{(0)}}\cdot -\frac{\chi_a^{2k_3}\Upsilon_g^{2k_4}\bar{\chi}_r^{2k_3}\bar{\Upsilon}_c^{2k_4}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_5}-\Omega_{k_6}}{\hbar}+i\eta^{(1)}}\cdot\frac{\chi_f^{1k_5}\bar{\chi}_d^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_4}+\Omega_{k_5}}{\hbar}+i\eta^{(2)}}\cdot -\frac{\Upsilon_b^{1k_6}\bar{\Upsilon}_h^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_4}}{\hbar}+i\eta^{(3)}}$

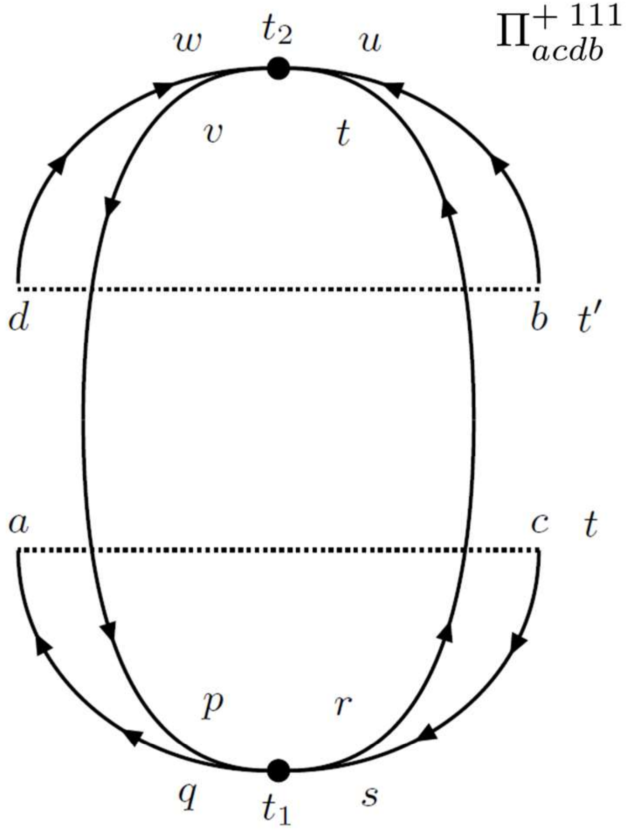
(3) $\frac{1}{8\hbar^3}\sum_{efgh}\sum_{pqrs}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\bar{V}_{pqrs}u^{11}_{ef}u^{11}_{gh}\cdot -\frac{\chi_p^{2k_1}\Upsilon_s^{1k_2}\bar{\chi}_q^{1k_1}\bar{\Upsilon}_e^{1k_2}}{-\frac{-\Omega_{k_2}+\Omega_{k_3}}{\hbar}+i\eta^{(0)}}\cdot -\frac{\chi_a^{2k_4}\Upsilon_b^{1k_3}\bar{\chi}_g^{1k_4}\bar{\Upsilon}_r^{2k_3}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_3}-\Omega_{k_5}}{\hbar}+i\eta^{(1)}}\cdot\frac{\chi_h^{1k_5}\bar{\chi}_d^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_4}}{\hbar}+i\eta^{(2)}}\cdot -\frac{\Upsilon_f^{1k_6}\bar{\Upsilon}_c^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_4}+\Omega_{k_6}}{\hbar}+i\eta^{(3)}}$

(4) $-\frac{1}{8\hbar^3}\sum_{efgh}\sum_{pqrs}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\bar{V}_{pqrs}u^{11}_{ef}u^{11}_{\bar{g}h}\cdot -\frac{\chi_p^{2k_1}\Upsilon_s^{1k_2}\bar{\chi}_q^{1k_1}\bar{\Upsilon}_e^{1k_2}}{-\frac{-\Omega_{k_2}+\Omega_{k_3}}{\hbar}+i\eta^{(0)}}\cdot -\frac{\chi_g^{2k_4}\Upsilon_b^{1k_3}\bar{\chi}_d^{2k_4}\bar{\Upsilon}_r^{2k_3}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_3}-\Omega_{k_4}}{\hbar}+i\eta^{(1)}}\cdot\frac{\chi_a^{2k_5}\bar{\chi}_h^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_5}}{\hbar}+i\eta^{(2)}}\cdot -\frac{\Upsilon_f^{1k_6}\bar{\Upsilon}_c^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_5}+\Omega_{k_6}}{\hbar}+i\eta^{(3)}}$

(5) $-\frac{1}{8\hbar^3}\sum_{efgh}\sum_{pqrs}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\bar{V}_{pqrs}u^{11}_{gh}u^{11}_{e\bar{f}}\cdot -\frac{\chi_p^{2k_1}\Upsilon_s^{1k_2}\bar{\chi}_q^{1k_1}\bar{\Upsilon}_e^{1k_2}}{-\frac{-\Omega_{k_2}+\Omega_{k_3}}{\hbar}+i\eta^{(0)}}\cdot -\frac{\chi_r^{1k_3}\Upsilon_b^{1k_4}\bar{\chi}_d^{2k_3}\bar{\Upsilon}_g^{1k_4}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_3}-\Omega_{k_4}}{\hbar}+i\eta^{(1)}}\cdot\frac{\chi_a^{2k_5}\bar{\chi}_f^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_6}}{\hbar}+i\eta^{(2)}}\cdot -\frac{\Upsilon_h^{1k_6}\bar{\Upsilon}_c^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_5}+\Omega_{k_6}}{\hbar}+i\eta^{(3)}}$

(6) $\frac{1}{8\hbar^3}\sum_{efgh}\sum_{pqrs}\sum_{k_1k_2k_3}\sum_{k_4k_5k_6}\bar{V}_{pqrs}u^{11}_{e\bar{f}}u^{11}_{\bar{g}h}\cdot -\frac{\chi_p^{2k_1}\Upsilon_s^{1k_2}\bar{\chi}_q^{1k_1}\bar{\Upsilon}_e^{1k_2}}{-\frac{-\Omega_{k_2}+\Omega_{k_3}}{\hbar}+i\eta^{(0)}}\cdot -\frac{\chi_r^{1k_3}\Upsilon_g^{2k_4}\bar{\chi}_d^{2k_3}\bar{\Upsilon}_c^{2k_4}}{\frac{\omega\hbar+2\Omega_0-\Omega_{k_3}-\Omega_{k_6}}{\hbar}+i\eta^{(1)}}\cdot\frac{\chi_a^{2k_5}\bar{\chi}_f^{2k_5}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_3}+\Omega_{k_4}}{\hbar}+i\eta^{(2)}}\cdot -\frac{\Upsilon_b^{1k_6}\bar{\Upsilon}_h^{2k_6}}{-\frac{-\omega\hbar-2\Omega_0+\Omega_{k_4}+\Omega_{k_5}}{\hbar}+i\eta^{(3)}}$

- **Example** of a *second-order* Goldstone graph contributing to $\Pi^{+\,1111}_{acdb}(\omega) \quad (t > t')$, corresponding to the time ordering $\quad t_2 > t > t' > t_1 \quad$ In time representation, it gives:



$$\Pi^{+\,1111}_{acdb}(t,t') = \ldots - \frac{i}{\hbar^2}\sum_{pqrs}\sum_{tuwv} \bar{v}_{pqrs}\bar{v}_{tuvw} G^{(0)\,11}_{aq}(t,t_1)$$
$$\cdot \quad G^{(0)\,11}_{sc}(t_1,t)\, G^{(0)\,11}_{vp}(t_2,t_1)\, G^{(0)\,11}_{rt}(t_1,t_2)\, G^{(0)\,11}_{wd}(t_2,t')$$
$$\cdot \quad G^{(0)\,11}_{bu}(t',t_2)\theta(t'-t_1)\theta(t-t')\theta(t_2-t) + \ldots$$

The presence of the sole *normal* propagators guarantees that

$\omega_{k_2}, \omega_{k_4}, \omega_{k_6} \Longrightarrow$ states with odd number of nucleons (largest exp. contrib. $= A-1$ state)

$\omega_{k_1}, \omega_{k_3}, \omega_{k_5} \Longrightarrow$ states with odd number of nucleons (largest exp. contrib. $= A+1$ state)

- In energy repr. this Goldstone graph translates into the following contribution to the second order transition function:

$$T^{(2)}(\omega) = \ldots - i \sum_{abcd}\sum_{pqrs}\sum_{tuwv}\sum_{\substack{k_1 k_2\\ k_3}}\sum_{\substack{k_4 k_5\\ k_6}} D^*_{ac}\bar{v}_{pqrs}\bar{v}_{tuvw} \frac{{}^{k_1}\chi^{(0)\,1}_{a}\; {}^{k_1}\Upsilon^{(0)\,1}_{q}\; {}^{k_2}\chi^{(0)\,1}_{c}\; {}^{k_2}\Upsilon^{(0)\,1}_{s}}{\omega_{k_{1,0}} + \omega_{k_{2,0}} + \omega_{k_{3,0}} + \omega_{k_{4,0}}}$$
$$\cdot \quad \frac{{}^{k_5}\chi^{(0)\,1}_{v}\; {}^{k_3}\Upsilon^{(0)\,1}_{p}\; {}^{k_4}\chi^{(0)\,1}_{t}\; {}^{k_4}\Upsilon^{(0)\,1}_{r}}{\omega - (\omega_{k_{1,0}} + \omega_{k_{2,0}} + \omega_{k_{3,0}} + \omega_{k_{4,0}} + \omega_{k_{5,0}} + \omega_{k_{6,0}})/\hbar} \frac{{}^{k_5}\chi^{(0)\,1}_{w}\; {}^{k_5}\Upsilon^{(0)\,1}_{d}\; {}^{k_6}\chi^{(0)\,1}_{u}\; {}^{k_6}\Upsilon^{(0)\,1}_{b}}{\omega_{k_{3,0}} + \omega_{k_{4,0}} + \omega_{k_{5,0}} + \omega_{k_{6,0}}} D_{db} + \ldots$$

Conclusion

- Motivated by the successes of SCGGF theory in the prediction of physical observables from the one-body propagator, we are extending the approach to quantities accessible from the polarization propagator, such as the excitation spectrum of even-even semi-magic nuclei and reduced EM multipole transition probabilities. In particular, we have
  - ✓ briefly recapitulated the state-of-the-art of Gorkov's SCGF theory;
  - ✓ defined the polarization propagator in Gorkov's formalism, in time and energy representation, together with its symmetry properties;
  - ✓ derived the self-consistent GBSE obeyed by Gorkov's polarization propagator, and displayed the components of the proper particle-hole vertex at first order;
  - ✓ introduced the *automated implementation of Wick's theorem* (AIWT) code for the perturbative expansion of the polarization propagator up to third order;
  - ✓ displayed examples in diagrammatic form of contributions up to third order in perturbation theory, generated automatically by the AIWT code;
  - ✓ shortly illustrated the ADC approach, its application to the irreducible self-energy and to the polariz. propagator, so far exploited in quantum chemistry;
  - ✓ shown examples of Goldstone diagrams, corresponding to expressions in energy repr. generated by the AIWT code, instrumental for the application of the ADC.

Thank you for the attention!

Appendix

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Ab initio Nuclear Many-body Problem

- Adopting realistic interactions, nuclei are described in terms of Z ***protons*** and N ***neutrons***, with the aim of

  understanding how nucleons organise themselves into nuclei (pairing, clustering ...)

  providing reliable predictions for nuclear observables (excited states, transitions ...)

Tool: the A-body Schrödinger equation $H\Psi_k^A = E_k^A \Psi_k^A$

where Ψ_k^A is the A-body wavefunction, associated with the energy eigenvalue E_k^A

- In H, realistic interactions are drawn from ***Chiral Effective Field Theory***, which provides

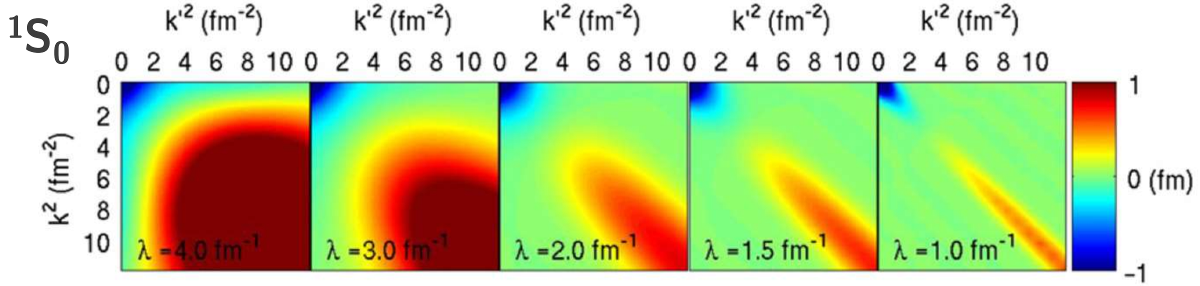
  a direct link with low-energy QCD and its symmetries

  a systematic framework to construct many-body interactions

  a theoretical error, stemming from the truncation of the expansion in powers of Q/Λ_χ

  where Λ_χ *is the chiral-symmetry-breaking scale* Q *is the 'small momentum' or pion mass*

In practice, ChEFT forces are preprocessed via the ***similarity renormalization group***, in order to quench the coupling between low and high momenta in the Hamiltonian



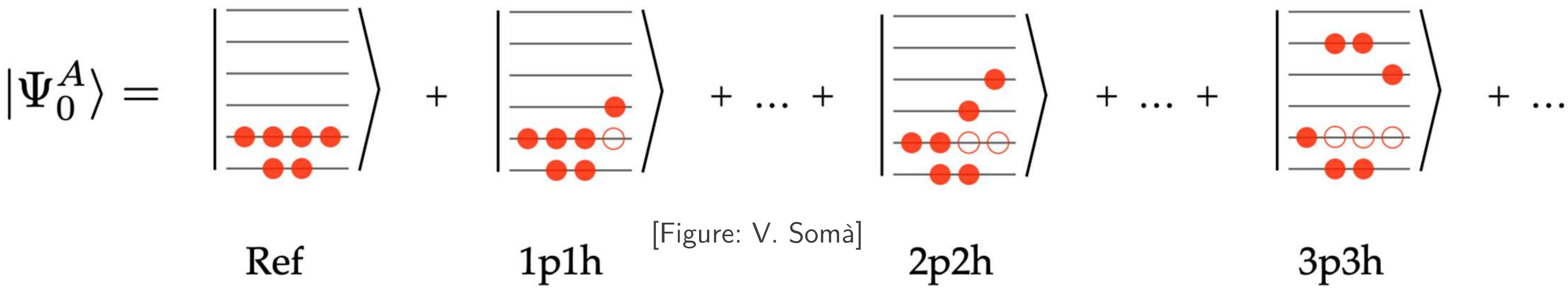
[S. Bogner et al. 2010]

Ab Initio Nuclear Wavefunction

Efficient approximation schemes for the nuclear wavefunction entail ***a polynomial scaling*** in the size M of the space of single-particle excitations $\rightsquigarrow M^{\alpha}$ with $\alpha \geq 4$

Correlation-expansion methods: expansion of the exact nuclear wavefunction into the space of particle-hole excitations built through the correlator Ω on a given *reference state*:

$$|\Psi_0^A\rangle = \Omega|\Phi_0^A\rangle = |\Phi_0^A\rangle + |\Phi_0^{A\,1p1h}\rangle + \ldots + |\Phi_0^{A\,2p2h}\rangle + \ldots + |\Phi_0^{A\,3p3h}\rangle + \ldots$$



[Figure: V. Somà]

where Ψ_0^A is the exact ground eigenstate of the A-body Hamiltonian, H

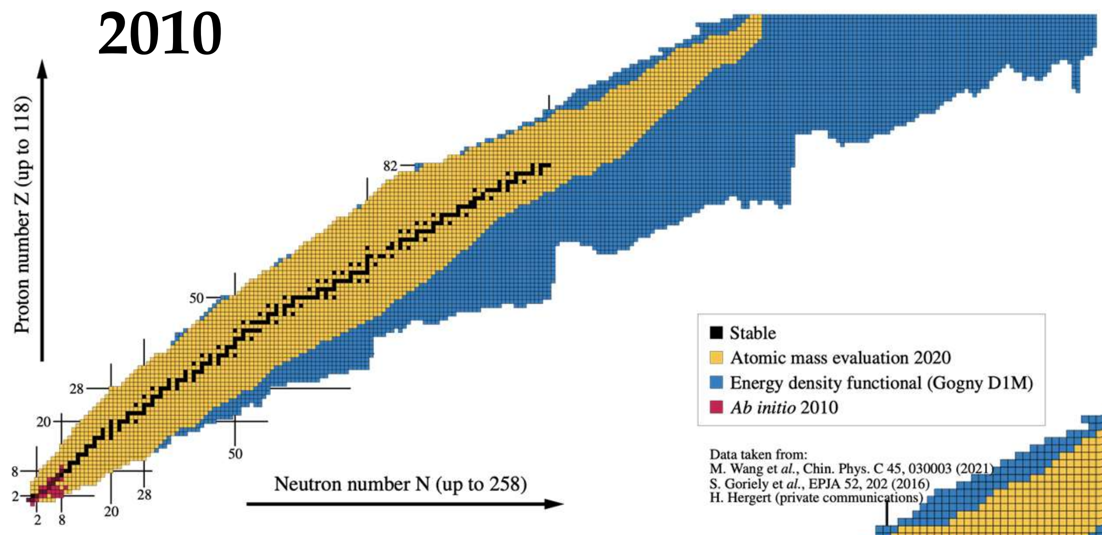
and the reference state Φ_0^A is the ground state of H_0, a solvable Hamiltonian, splitting the original one into $H = H_0 + H_I$ where H_I contains the 2-, 3-, ... -body interactions

Problem: *In open-shell nuclei, the ground state is almost degenerate with respect to the excitation of pairs of nucleons in the same single-particle energy level*

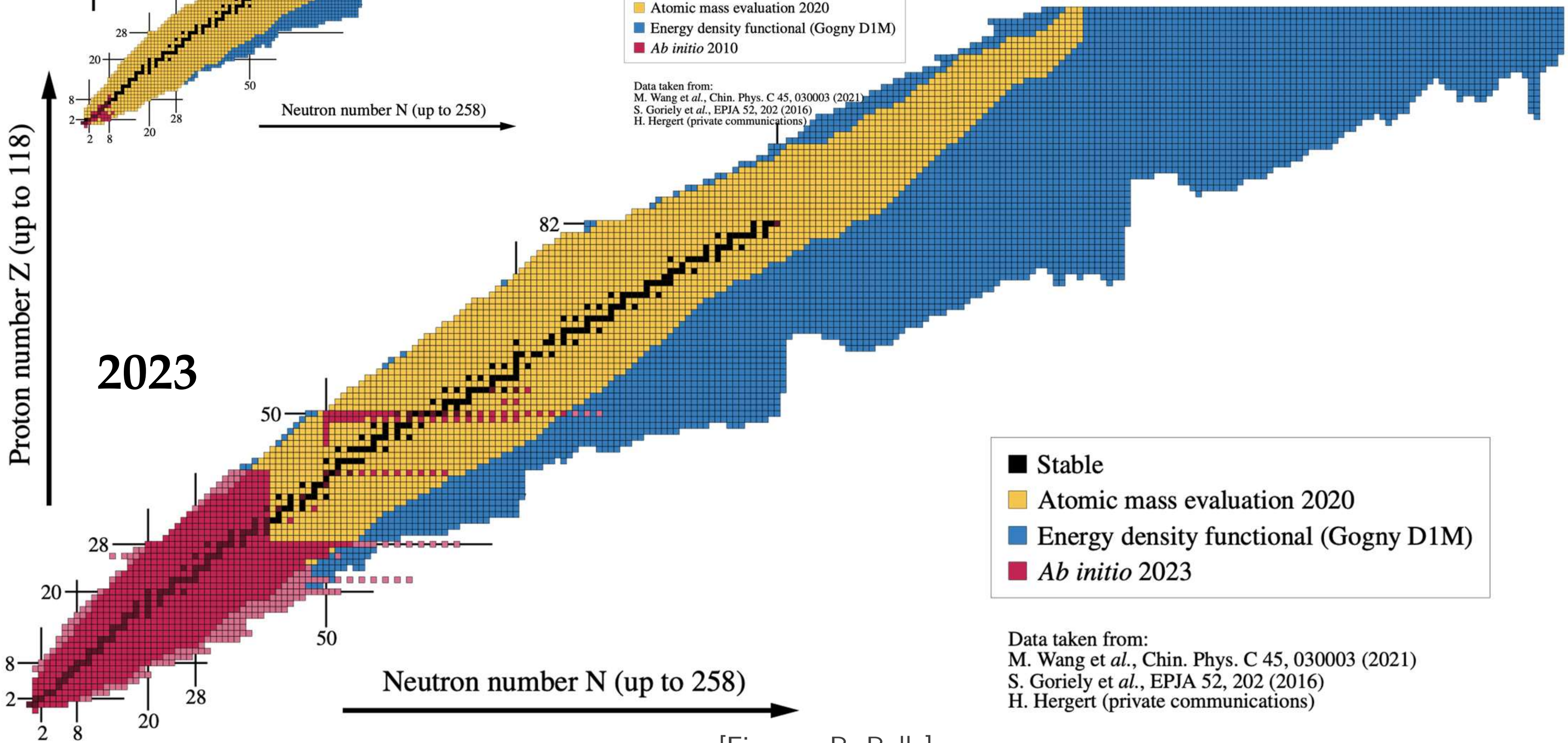
$\Downarrow$ $\Downarrow$ $\Downarrow$

Solution: in the reference state, breaking the symmetry associated to **particle number**, (semi-magic nuclei) together with **rotational symmetry** (doubly-open-shell nuclei)

Ab initio Nuclear Chart



- A very significant **increase** in the reach of *ab-initio approaches* took place in the last decade...



[Figures: B. Bally]

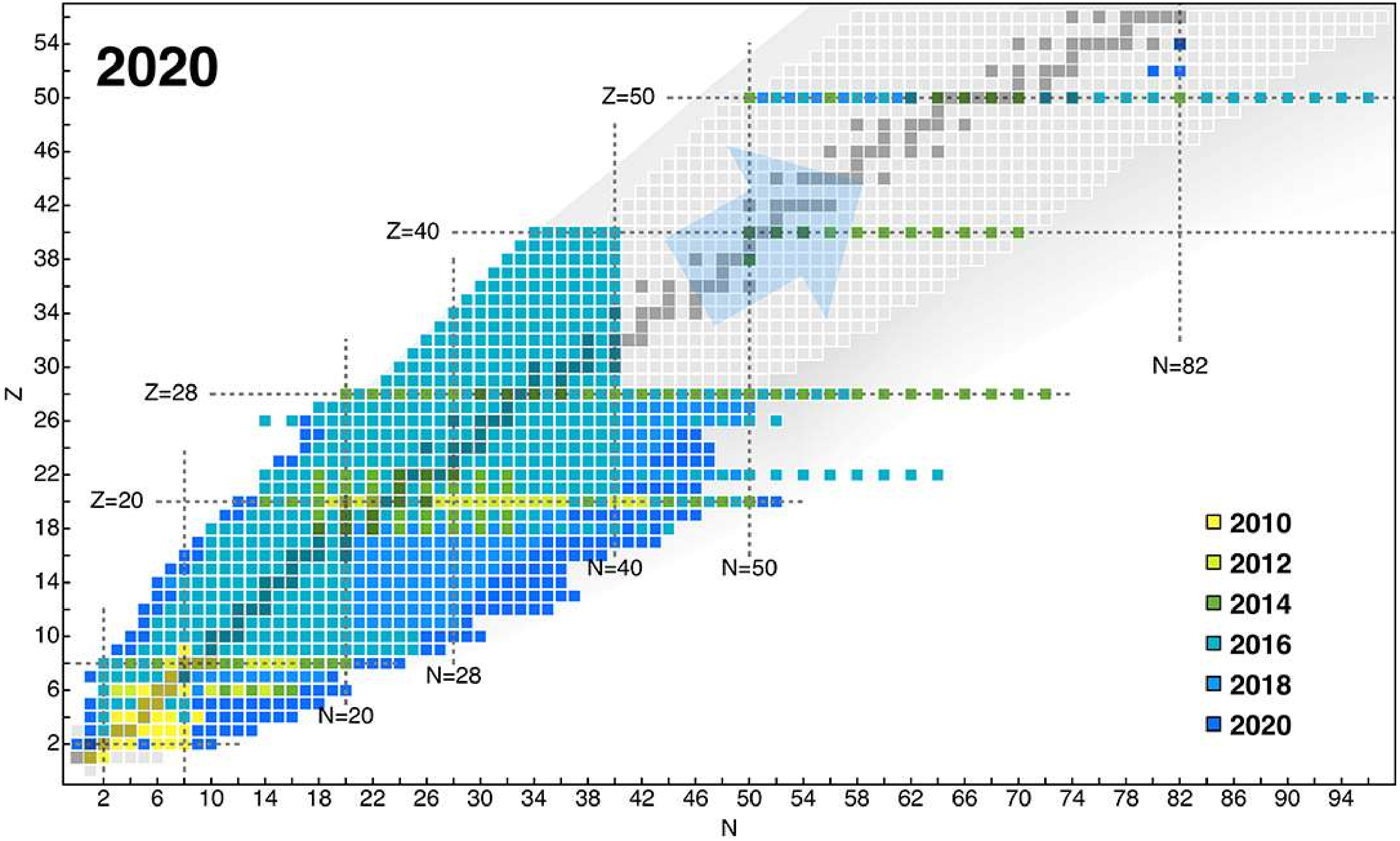
Main approaches:

Magic nuclei: MBPT, SCGF, IMSRG, CI, CC ...

Semi-magic nuclei: MR-IMSRG, BMBPT, SCGGF, BCC, ...

Doubly open-shell nuclei: MR-IMSRG, BMBPT, SCGGF$^+$, CC, ...

Passepartout: FCI, NCSM, NLEFT, LQCD ($A < 4$), PGCM-PT ...



[Figure: H. Hergert]

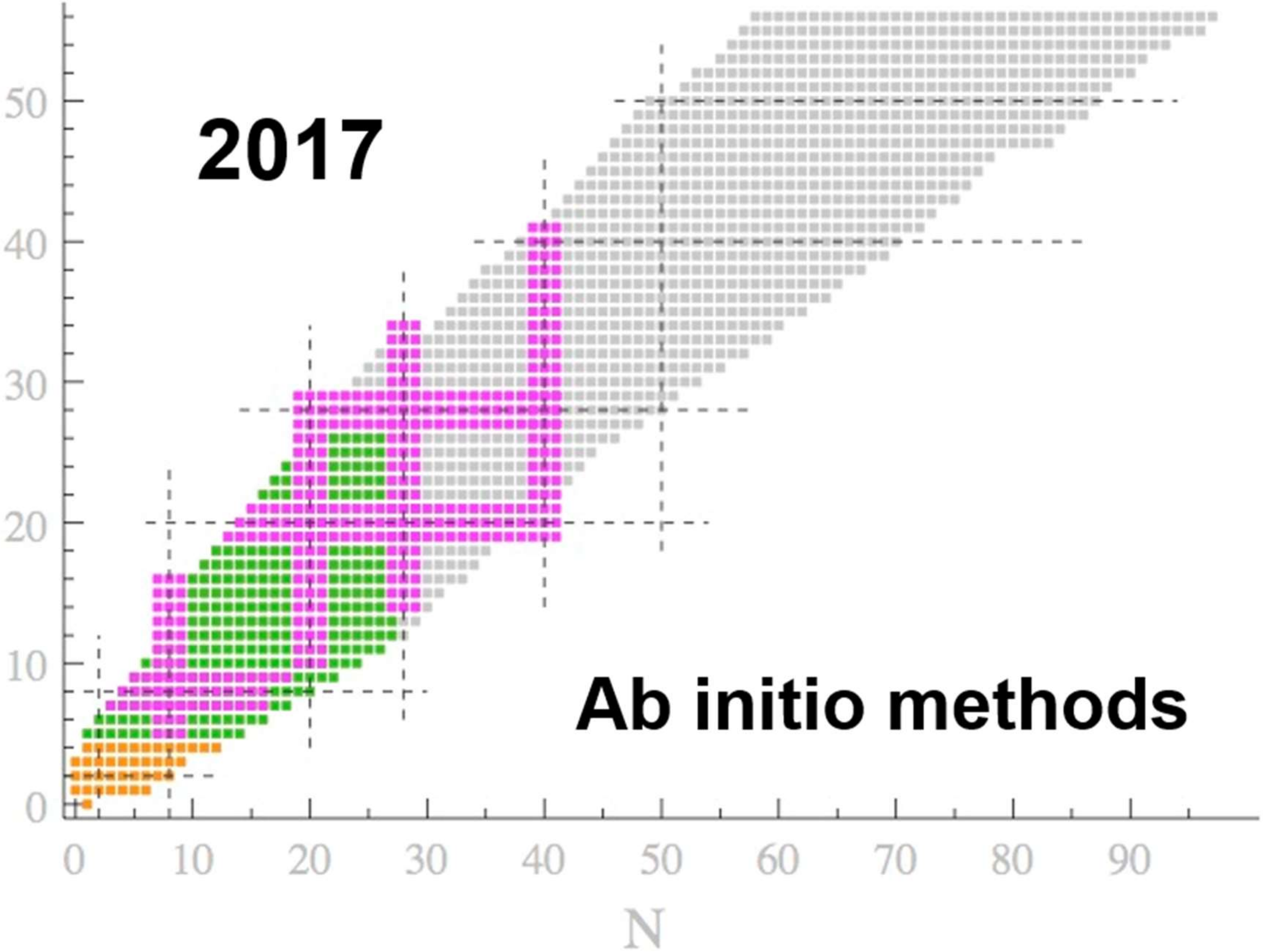
Ab initio models of Nuclear Structure

Exact solution of the Schr. equation: exponential or factorial scaling with the system size (A)
Approximate solution: polynomial scaling with A in the correlation expansion methods

Approximate solution for magic and semi-magic nuclei with A > 11.
Tools: MBPT/BMBPT, SCGF/SCGGF, IMSRG, CC/BCC ...

Approximate solution for open-shell nuclei with A > 11
Tools: BMBPT, CI, BCC, MR-IMSRG...

Exact solution for nuclei with A < 12
Tools: MBPT, NLEFT, NCSM, LQCD (A < 4)...



NAMBU COVARIANT PERTURBATION THEORY

We adopt the formalism of Nambu-covariant perturbation theory (NCPT) M. Drissi et al, arXiv:2107.09763

Purpose: extension of the SCGF approach to tackle the near-degeneracy of the ground states of singly open-shell nuclei with respect to creation/annihilation of pairs of nucleons with opposite j_z

Duplication of the Hilbert space associated to a single-nucleon $\mathscr{H}_1^e \equiv \mathscr{H}_1 \otimes \mathscr{H}_1^\dagger$
where $\mathcal{B} \subset \mathscr{H}_1$ is a basis and $\bar{\mathcal{B}} \subset \mathscr{H}_1^\dagger$ its dual and $|b\rangle,\ |\bar{b}\rangle \in \mathcal{B},\ \langle b|,\ \langle \bar{b}| \in \mathcal{B}^\dagger$

- Second quantization operators: $a_b,\ a_{\bar{b}}$ and $a_b^\dagger,\ a_{\bar{b}}^\dagger$

  where the *involution* (s.p. space) is defined: $a_{\bar{b}} = \eta_b a_{\tilde{b}} \quad a_{\bar{b}}^\dagger = \eta_b a_{\tilde{b}}^\dagger$ with $\tilde{b} \equiv (n, \ell, j, -m, q)$, $b \equiv (n, \ell, j, m, q)$ where $\eta_b = (-1)^{\ell - j - m}$, $\eta_b \eta_b^* = \eta_b^2 = 1$, $\eta_b \eta_{\tilde{b}} = -1$

  ...which are grouped into *Nambu* vectors:

$$\bar{B}_{(b,1)} \equiv a_b^\dagger \qquad \bar{B}_{(b,2)} \equiv \eta_b a_{\tilde{b}} = a_{\bar{b}}$$

$$B^{(b,1)} \equiv a_b \qquad B^{(b,2)} \equiv \eta_b a_{\tilde{b}}^\dagger = a_{\bar{b}}^\dagger$$

...and $l = 1, 2$ are Nambu indices.

- The canonical anticommutation rules

$$\left\{\bar{B}_\mu, \bar{B}_\nu\right\} = g_{\mu\nu} \qquad \left\{\bar{B}_\mu, B^\nu\right\} = g_\mu{}^\nu \qquad \left\{B^\mu, \bar{B}_\nu\right\} = g^\mu{}_\nu \qquad \left\{B^\mu, B^\nu\right\} = g^{\mu\nu}$$

define the elements of the *metric tensor*:

$$g^{\alpha\beta} \equiv \delta_{a\tilde{b}} \delta_{l_a \bar{l}_b} [\delta_{1 l_a} \eta_{\tilde{a}} + \delta_{2 l_a} \eta_a] \qquad g_{\alpha\beta} \equiv \delta_{a\tilde{b}} \delta_{l_a \bar{l}_b} [\delta_{1 l_a} \eta_{\tilde{a}} + \delta_{2 l_a} \eta_a] \qquad g^\alpha{}_\beta = g^{\alpha\gamma} g_{\gamma\beta} = \delta_{ab} \delta_{l_a l_b}$$

$g^{\alpha\beta} \wedge g_{\alpha\beta}$ are antidiagonal in both the Nambu and the s.p. space! $g_\alpha{}^\beta = g_{\alpha\gamma} g^{\gamma\beta} = \delta_{ab} \delta_{l_a l_b}$

Theoretical Framework

- **Method**: the degeneracy wrt *ph*-excitations is lifted via the *Bogoliubov reference state* and transferred into a degeneracy wrt the operations of the symmetry group $U_Z(1) \times U_N(1)$

$$\Omega_0^{A+2}(\mathrm{Z}+2,\mathrm{N}) \approx \Omega_0^{A}(\mathrm{Z},\mathrm{N}) \implies E_0^{Z+2}(\mathrm{Z}+2,\mathrm{N}) - E_0^{A}(\mathrm{Z},\mathrm{N}) \approx E_0^{A}(\mathrm{Z},\mathrm{N}) - E_0^{A-2}(\mathrm{Z}-2,\mathrm{N}) \approx \ldots \approx 2\mu_p$$

$$\Omega_0^{A+2}(\mathrm{Z},\mathrm{N}+2) \approx \Omega_0^{A}(\mathrm{Z},\mathrm{N}) \implies E_0^{A+2}(\mathrm{Z},\mathrm{N}+2) - E_0^{A}(\mathrm{Z},\mathrm{N}) \approx E_0^{A}(\mathrm{Z},\mathrm{N}) - E_0^{A-2}(\mathrm{Z},\mathrm{N}-2) \approx \ldots \approx 2\mu_n$$

the constituents can be added or removed almost *at the same energy cost*, irrespective of A.

- **Observation**: The choice of *U* corresponds to selecting a superfluid unperturbed $g.s.$, acting as reference for the application of *Wick's theorem*. The exact eigenstates of Ω, preserve A:

$$H|\Psi_0^A\rangle = E_0^A|\Psi_0^A\rangle \qquad \Omega|\Psi_0^A\rangle = (E_0^A - \mu_p\mathrm{Z} - \mu_n\mathrm{N})|\Psi_0^A\rangle \equiv \Omega_0^A|\Psi_0^A\rangle$$

Considering the superposition of the g.s. of the nuclear systems with even number of constituents

$$|\Psi_0^{\mathrm{SB}}\rangle = \sum_{\mathrm{n}}^{\mathrm{even}} c_{\mathrm{n}}|\Psi_0^{\mathrm{n}}\rangle \qquad \text{one **replaces**} \qquad |\Psi_0\rangle \equiv |\Psi_0^A\rangle \quad \text{with} \quad |\Psi_0^{\mathrm{SB}}\rangle$$

where the coefficients of the expansion in the Fock space minimize:

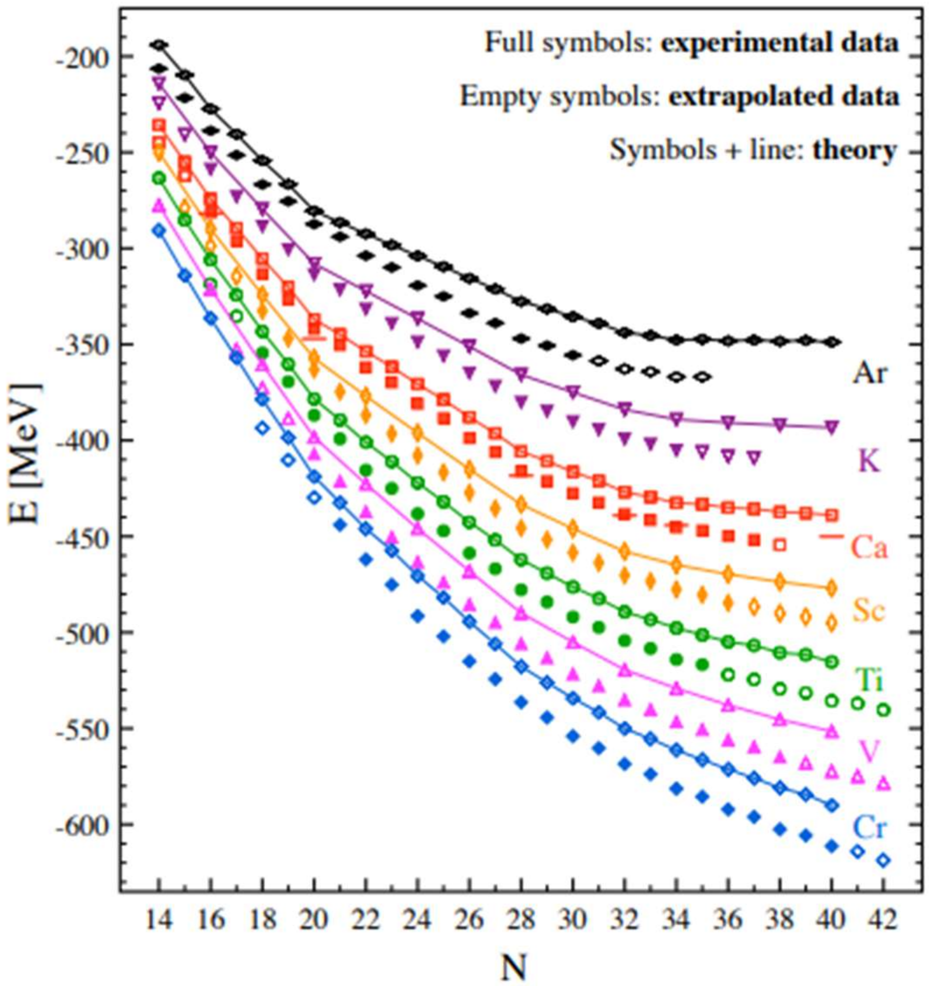
$$\Omega_0^{\mathrm{SB}} \equiv \langle\Psi_0^{\mathrm{SB}}|\Omega|\Psi_0^{\mathrm{SB}}\rangle \gtrsim \Omega_0^A$$

subject to three constraints:

$$\mathrm{Z} = \langle\Psi_0^{\mathrm{SB}}|Z|\Psi_0^{\mathrm{SB}}\rangle \qquad \mathrm{N} = \langle\Psi_0^{\mathrm{SB}}|N|\Psi_0^{\mathrm{SB}}\rangle \qquad \langle\Psi_0^{\mathrm{SB}}|\Psi_0^{\mathrm{SB}}\rangle = \sum_{\mathrm{n}}^{\mathrm{even}} |c_{\mathrm{n}}|^2 = 1$$

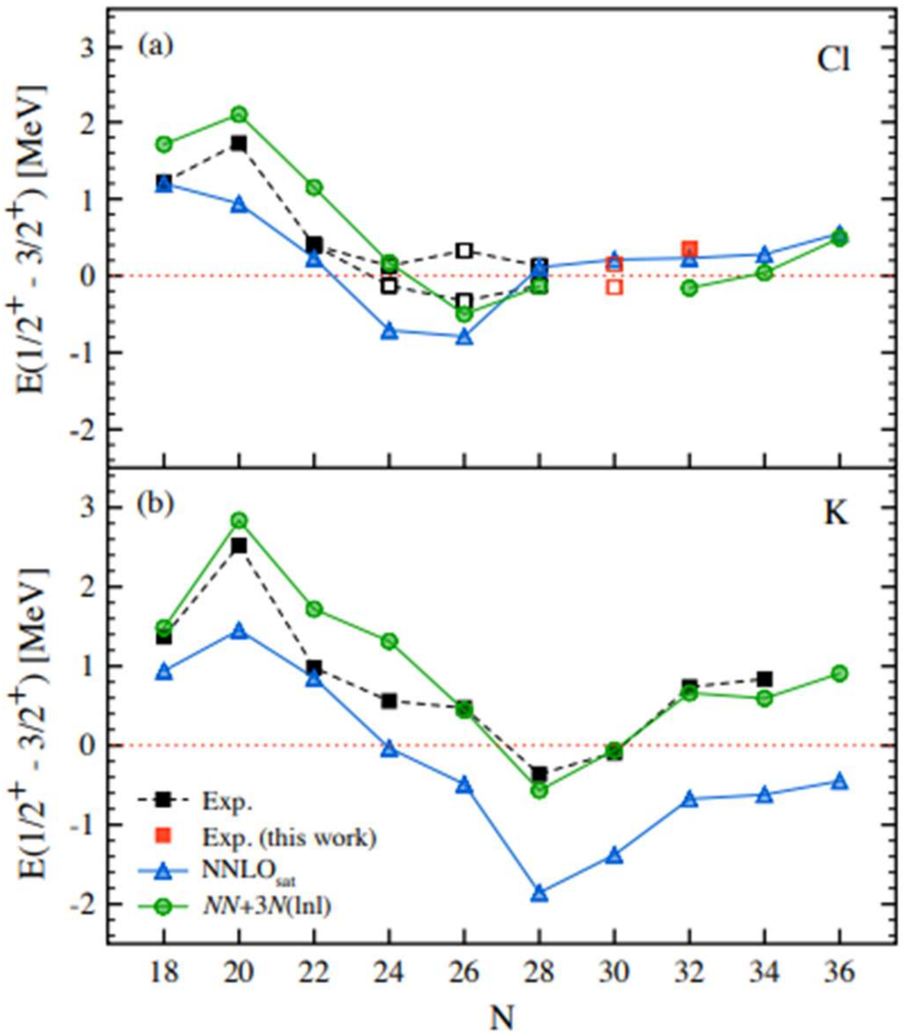
APPENDIX
PHYSICAL OBSERVABLES
from the one-body propagator

- Binding energy of even-even isotopic chains: the $18 \leq Z \leq 24$ nuclei



[*Eur. Phys. J A* **57**, 135 (2021)]

- Energies of the excited states of odd-even systems: the first $1/2^+$ and $3/2^+$ levels (Cl & K)



[*Phys. Rev. C* ***104***, 044331 (2021)]

PHYSICAL OBSERVABLES
from the one-body propagator

- Gorkov *spectral functions*:

$$\mathbf{S}^{+}_{ab}(\omega) = -\frac{1}{\pi}\mathfrak{Im}\ \mathbf{G}_{ab}(\omega) = \sum_k {}^k\boldsymbol{\chi}_a\ {}^k\boldsymbol{\chi}^{\dagger}_b \delta(\omega - \omega_k)$$

with $\omega > 0$

$$\mathbf{S}^{-}_{ab}(\omega) = +\frac{1}{\pi}\mathfrak{Im}\ \mathbf{G}_{ab}(\omega) = \sum_k {}^k\boldsymbol{\Upsilon}_a\ {}^k\boldsymbol{\Upsilon}^{\dagger}_b \delta(\omega + \omega_k)$$

with $\omega < 0$

From the normal components, one nucleon removal and addition amplitudes are extracted:

$$S^{h}_{ab}(\omega) \equiv S^{11}_{ab}(\omega)$$
$$S^{p}_{ab}(\omega) \equiv S^{+}_{ab}(\omega)$$

- One and two-neutron separation energies:

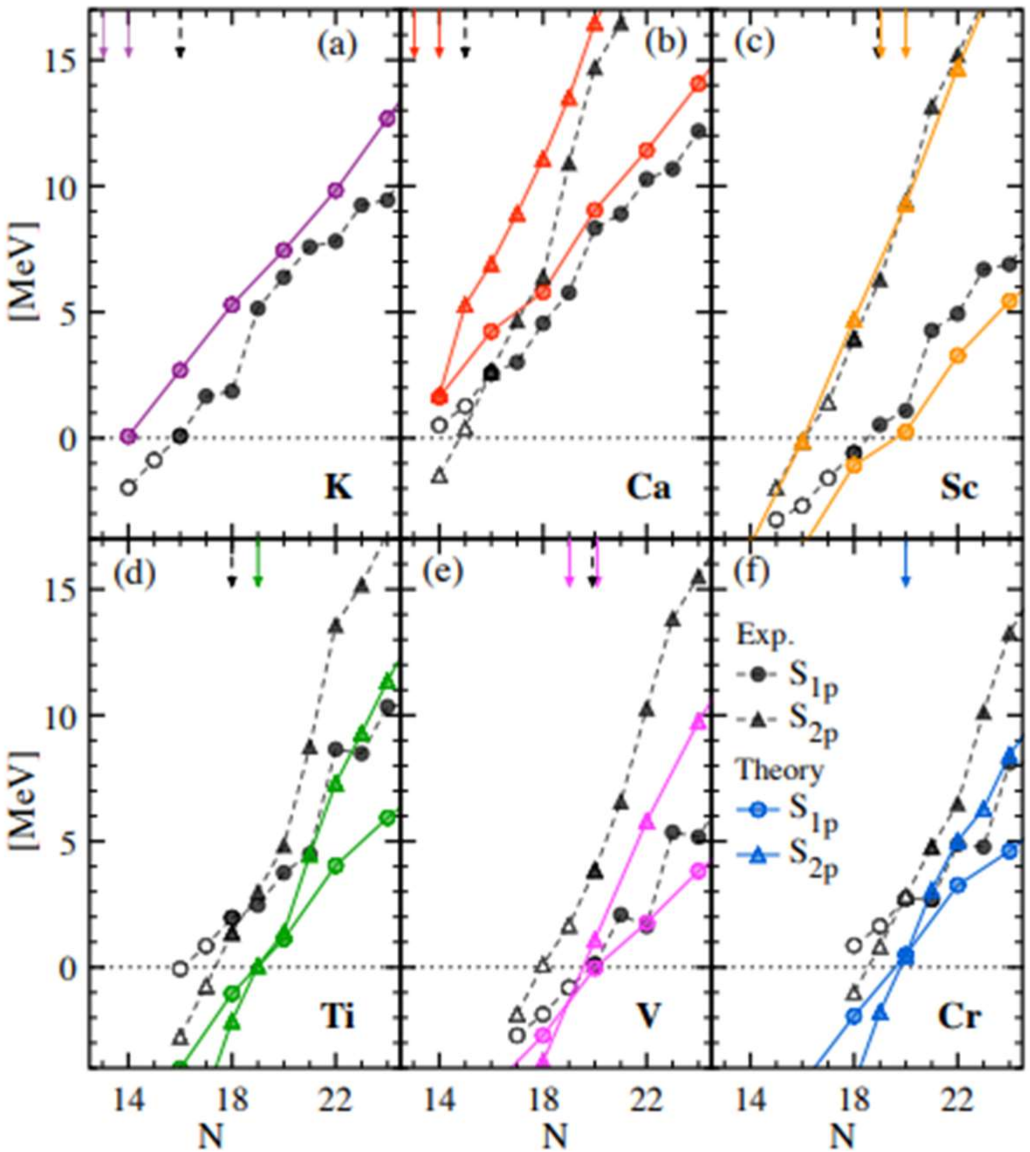
$$S_n(N,Z) \equiv |E(N,Z)| - |E(N-1,Z)|$$
$$S_{2n}(N,Z) \equiv |E(N,Z)| - |E(N-2,Z)|$$

- One and two-proton separation energies:

$$S_p(N,Z) \equiv |E(N,Z)| - |E(N,Z-1)|$$
$$S_{2p}(N,Z) \equiv |E(N,Z)| - |E(N,Z-2)|$$

where $E(N,Z) \rightsquigarrow$ g.s. energy

$S_p(N,Z)$ and $S_{2p}(N,Z)$



[*Eur. Phys. J A* **57**, 135 (2021)]

Time Evolution of Operators and States

- **Heisenberg's picture**: time evolution of Nambu second-quantization operators follows

$$\begin{aligned} \mathbf{A}_b(t) = \mathbf{A}_{\Omega\, b}(t) &\equiv e^{i\Omega t/\hbar}\mathbf{A}_b e^{-i\Omega t/\hbar} \\ \mathbf{A}_b^\dagger(t) = [\mathbf{A}_{\Omega\, b}(t)]^\dagger &\equiv e^{i\Omega t/\hbar}\mathbf{A}_b^\dagger e^{-i\Omega t/\hbar} \end{aligned}$$

As in standard Heisenberg's picture, the states are time-independent:

$$|\Psi_0\rangle \equiv |\Psi_0(t)\rangle = |\Psi_0(t_0)\rangle \quad \forall\ t, t_0$$

- **Interaction picture**: time evolution of Nambu second-quantization operators follows

$$\begin{aligned} \mathbf{A}_{I\, b}(t) &\equiv e^{i\Omega_U t/\hbar}\mathbf{A}_b e^{-i\Omega_U t/\hbar} \\ [\mathbf{A}_{I\, b}(t)]^\dagger &\equiv e^{i\Omega_U t/\hbar}\mathbf{A}_b^\dagger e^{-i\Omega_U t/\hbar} \end{aligned}$$

States evolve as in the standard interaction picture:

$$|\Psi_{I\,0}\rangle \equiv e^{i\Omega_U/\hbar} e^{-i\Omega/\hbar}|\Psi_0\rangle$$

- **Field picture**: time evolution of Nambu second-quantization operators follows

$$\begin{aligned} \mathbf{A}_{F\, b}(t) &\equiv e^{i\Omega t/\hbar}\mathbf{A}_b e^{-i\Omega t/\hbar} \\ [\mathbf{A}_{F\, b}(t)]^\dagger &\equiv e^{i\Omega t/\hbar}\mathbf{A}_b^\dagger e^{-i\Omega t/\hbar} \end{aligned}$$

where $\Omega_I^\phi = \Omega_I + \phi$ contains the ext. field $\phi(t)$ and the states evolve as

$$|\Psi_{F\,0}\rangle \equiv e^{i\Omega/\hbar} U_S^\phi(t,0)|\Psi_0\rangle$$

and $U_S^\phi(t,0)$ is Schrödinger's time evolution operator wrt the grand canonical potential $\Omega_U + \Omega_I^\phi$

GORKOV'S EQUATIONS

▶ The one-body Gorkov-Green's functions obey the following generalization of Dyson's equation:

$$G^{\alpha}{}_{\beta}(\omega) = G^{(0)\alpha}{}_{\beta}(\omega) + \sum_{\gamma\delta} G^{(0)\alpha}{}_{\gamma}(\omega)\tilde{\Sigma}^{\gamma}{}_{\delta}(\omega)G^{\delta}{}_{\beta}(\omega)$$

where the self-energy can be subdivided into a proper part and a contribution from the aux. potential

$$\tilde{\Sigma}^{\alpha}{}_{\beta}(\omega) \equiv \Sigma^{\alpha}{}_{\beta}(\omega) - U^{\alpha}{}_{\beta} - \tilde{U}^{\alpha}{}_{\beta}$$

and $G^{(0)\alpha}{}_{\beta}(\omega)$ are the unperturbed propagators.

Since U acts as a mean field, the Hartree-Fock-Bogoliubov (HFB) one-body propagators, solution of the problem $\Omega_U = \Omega_{\text{HFB}}$ can be exploited for $G^{(0)\alpha}{}_{\beta}(\omega)$ as well as an input for $G^{\alpha}{}_{\beta}(\omega)$ at the r.h.s. of the self-consistent equation. The numerical $G^{\alpha}{}_{\beta}(\omega)$ are obtained though ***BcDor* codes**

- *in practice*: energy-independent self-consistent equations for $G^{\alpha}{}_{\beta}(\omega)$ are solved.

EXAMPLES (proper self-energy to first order):

$$\Sigma^{(a,1)}{}_{(b,1)}(\omega)\Big|_1 \equiv -i \sum_{c,l_c}\sum_{d,l_d}\int_{C\uparrow}\frac{d\omega'}{2\pi}\frac{1}{3!} \times V^{(a,1)(c,l_c)}{}_{(d,l_d)(b,1)} G^{(d,l_d)}{}_{(c,l_c)}(\omega)$$

$$\Sigma^{(a,2)}{}_{(b,2)}(\omega)\Big|_1 \equiv -i \sum_{c,l_c}\sum_{d,l_d}\int_{C\downarrow}\frac{d\omega'}{2\pi}\frac{1}{3!} \times V_{(b,2)(d,l_d)}{}^{(c,l_c)(a,2)} G^{(d,l_d)}{}_{(c,l_c)}(\omega)$$

$$\Sigma^{(a,1)}{}_{(b,2)}(\omega)\Big|_1 \equiv -i \sum_{c,l_c}\sum_{d,l_d}\int_{C\uparrow}\frac{d\omega'}{2\pi}\frac{1}{3!} \times V^{(a,1)}{}_{(b,2)}{}^{(d,l_d)}{}_{(c,l_c)} G^{(c,l_c)}{}_{(d,l_d)}(\omega)$$

$$\Sigma^{(a,2)}{}_{(b,1)}(\omega)\Big|_1 \equiv -i \sum_{c,l_c}\sum_{d,l_d}\int_{C\uparrow}\frac{d\omega'}{2\pi}\frac{1}{3!} \times V_{(c,l_c)}{}^{(d,l_d)}{}_{(b,1)}{}^{(a,2)} G^{(c,l_c)}{}_{(d,l_d)}(\omega)$$

Dyson's Polarization Propagator

- In SCGF theory, the *polarization propagator* is obtained from the two-body response function.

  Adopting the covention of J. Schirmer, *Phys. Rev. A* **26**, 5, 2395-2416 (1982), the latter reads:

$$\mathcal{R}^{(A,A)}_{abcd}(t,t';t'',t''') \equiv \mathcal{G}^{(A,A)}_{abcd}(t,t',t'',t''') - \mathcal{G}^{(A,A)}_{ac}(t,t'')\mathcal{G}^{(A,A)}_{bd}(t',t''')$$

where $$\mathcal{G}^{(A,A)}_{abcd}(t,t',t'',t''') \equiv (-i)^2\langle\Psi^A_0|T\Big\{a_a(t)a_b(t')a^\dagger_d(t''')a^\dagger_c(t'')\Big\}|\Psi^A_0\rangle$$

... is the two-body Green's function.

and $$\mathcal{G}^{(A,A)}_{ab}(t,t') \equiv (-i)\langle\Psi^A_0|T\Big\{a_a(t)a^\dagger_b(t')\Big\}|\Psi^A_0\rangle$$

... is the one-body Green's function.

- Taking the two-time limit the *polarization propagator* is obtained:

$$\Pi_{acdb}(t,t') = \lim_{\substack{t''\to t^+ \\ t'''\to t'^+}} i\mathcal{R}^{(A,A)}_{abcd}(t,t';t'',t''')$$

alternatively, the limits $t''\to t'^+ \ \wedge \ t'''\to t''^+$ can be considered. In the first case, one writes

$$\Pi_{acdb}(t,t') = -i\langle\Psi^A_0|T\Big\{a^\dagger_c(t)a_a(t)a^\dagger_d(t')a_b(t')\Big\}|\Psi^A_0\rangle$$
$$+i\langle\Psi^A_0|T\Big\{a^\dagger_c(t)a_a(t)\Big\}|\Psi^A_0\rangle\langle\Psi^A_0|T\Big\{a^\dagger_d(t')a_b(t')\Big\}|\Psi^A_0\rangle$$

Dyson's Polarization Propagator

If the Schrödinger problem is time-independent, the Fourier transform is function of one frequency

$$\Pi_{acdb}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}(t-t') e^{i\omega(t-t')} \Pi_{acdb}(t,t')$$

The ensuing Lehmann representation can be decomposed into two interrelated parts,

$$\Pi_{acdb}(\omega) = \Pi^{+}_{acdb}(\omega) + \Pi^{-}_{acdb}(\omega)$$

analytical in the upper part of the complex plane ... ... and in the lower one.

$$\Pi^{+}_{acdb}(\omega) \equiv \sum_{k \neq 0} \frac{\langle \Psi_0^A | a_c^\dagger a_a | \Psi_k^A \rangle \langle \Psi_k^A | a_d^\dagger a_b | \Psi_0^A \rangle}{\omega - (E_k^A - E_0^A) + i\eta} \qquad \Pi^{-}_{acdb}(\omega) = -\sum_{k \neq 0} \frac{\langle \Psi_0^A | a_d^\dagger a_b | \Psi_k^A \rangle \langle \Psi_k^A | a_c^\dagger a_a | \Psi_0^A \rangle}{\omega + (E_k^A - E_0^A) - i\eta}$$

The relation between the two reads:

$${\Pi^{+}_{cabd}}^{*}(-\omega) = \Pi^{-}_{acdb}(\omega)$$

- Symmetry relations:

time reversal of H

$$\Pi_{acdb}(\omega) = \Pi_{bdca}(-\omega)$$

complex-conjugation

$$\Pi_{acdb}(\omega) = -\Pi^{*}_{dbac}(-\omega)$$

- The poles coincide with the energy of the *excited states* of the even-even system wrt the g.s.
- The residues of the poles are proportional to the *transition matrix elements*:

$${}^{k}X_{db} \equiv \langle \Psi_0^A | a_d^\dagger a_b | \Psi_k^A \rangle \qquad {}^{k}Y_{ca} \equiv \langle \Psi_k^A | a_c^\dagger a_a | \Psi_0^A \rangle$$

- Transition mediated by a one-body operator:

$$\langle \Psi_p^A | \mathcal{O} | \Psi_0^A \rangle = \sum_{ab} (a|\mathcal{O}|b) \langle \Psi_p^A | a_a^\dagger a_b | \Psi_0^A \rangle$$

Dyson's Polarization Propagator

Approximation methods

In SCGF theory, the determination of the polarization propagator in Lehmann representation may follow three different paths:

- the **direct** approach: the ADC scheme applied directly to the polarization propagator

J. Schirmer, *Phys. Rev. A* ***26***, 5, 2395-2416 (1982),

approx. scheme for the time-ordered diagrams contributing to the trans. function, linked to Π via a unitary transf.

...so far adopted in *molecular systems* (quantum chemistry): *Comput. Mol. Sci.* **5**, 82-95 (2015)

ADC(2)

Adv. Chem. Phys ***69***, *22*, 201-240 (1987)

J. of Chem. Phys **112**, *22*, 4173-4185 (2000)

ADC(3)

J. of Chem. Phys **111**, 9982-9999 (1999)

J. of Chem. Phys **117**, 6402-6409 (2002)

- the **self-consistent** (SC) approach: possible application of the ADC scheme on the interaction kernel K_{fegh}. In time repr. the SC equation for the *three-time* polarization prop. reads

$$\Pi_{acdb}(t,t',t'',t''^{+}) = \Pi^{(0)}_{acdb}(t,t',t'',t''^{+}) + \frac{i}{\hbar}\sum_{efgh}\int \mathrm{d}t_1 \int \mathrm{d}t_2 \int \mathrm{d}t_2 \int \mathrm{d}t_3 \Pi^{(0)}_{acef}(t,t',t_1,t_2)$$
$$\times K_{fegh}(t_2,t_1,t_3,t_4)\Pi_{ghdb}(t_3,t_4,t'',t''^{+})$$

Tool: the SC equation for the *two-time* polarization propagator in energy representation

W. Czyz, *Acta. Phys. Polonica* ***20***, 737 (1961).

- the **random phase approximation**: although self-consistent, it neglects interactions betw.

$$\Pi_{acdb}(\omega) = \Pi^{(0)}_{acdb}(\omega) + \Pi^{(0)}_{acef}(\omega)\bar{V}_{ehfg}\Pi_{ghdb}(\omega)$$

particles/holes propagating in different 'bubbles'. It is widely applied also in *nuclear systems*.

Algebraic Diagrammatic Construction

for the one-body propagator

It is an approximation scheme deveoped for the *polarization propagator* (J. Schirmer, *Phys. Rev. A* **26**, 5, 2395-2416 (1982)) and the *one-body propagator* (J. Schirmer, *Phys. Rev. A* **28**, 3, 1237-1259 (1983)) in SCGF theory. At present, only the extension to Gorkov's one-body propagators is operational.

Motivation: the ADC scheme permits to rewrite Gorkov's equations (in energy repr.) as an energy-independent eigenvalue problem, preserving the analytic structure of the self-energy.

⇝ V. Somà et al. *Phys. Rev. C 84, 064317 (2011)*

- Splitting of the *proper* self-energy into a static and a dynamic part:

$$\tilde{\mathbf{\Sigma}}_{ab}(\omega) = -\mathbf{U}_{ab} + \mathbf{\Sigma}^{(\text{stat})}_{ab} + \mathbf{\Sigma}^{(\text{dyn})}_{ab}$$

whose structure is

$$\mathbf{\Sigma}^{(\text{dyn})}_{ab}(\omega) = \mathbf{\Sigma}^{(\text{dyn})\,+}_{ab} + \mathbf{\Sigma}^{(\text{dyn})\,-}_{ab} = \sum_k \left[\frac{{}^k\mathbf{M}_a\,{}^k\mathbf{M}_b^\dagger}{\omega - \Omega_k/\hbar + i\eta} + \frac{{}^k\mathbf{N}_a\,{}^k\mathbf{N}_b^\dagger}{\omega + \Omega_k/\hbar - i\eta} \right]$$

It is sufficient to consider only $\mathbf{\Sigma}^{(\text{dyn})\,+}_{ab} \equiv \mathbf{M}_a(\mathbb{1}\omega - \mathbf{E})\mathbf{M}_b^\dagger$

- The ADC scheme postulates $\mathbf{\Sigma}^{(\text{dyn})\,+}_{ab} \overset{\text{ADC}}{=} \mathbf{C}_a(\omega\mathbb{1} - \mathbf{W} - \mathbf{P})^{-1}\mathbf{C}_b^\dagger$

where the matrices $\mathbf{C}_a$ and $\mathbf{P}$ in Nambu and k-space are expanded oder by order

$\mathbf{C}_a \equiv \mathbf{C}_a^{(1)} + \mathbf{C}_a^{(2)} + \dots$ $\quad \mathbf{P} \equiv \mathbf{P}^{(1)} + \mathbf{P}^{(2)} + \dots$ $\quad \mathbf{W} \Rightarrow$ Matrix of the unperturbed eigenvalues (Ω_U)

By exploiting the geometric series, the ADC ansatz can be rewritten as

$$\mathbf{\Sigma}^{(\text{dyn})\,+}_{ab} \overset{\text{ADC}}{=} \mathbf{C}_a(\omega\mathbb{1} - \mathbf{W})^{-1} \sum_{n=0}^{+\infty} \left\{ \mathbf{P}(\omega\mathbb{1} - \mathbf{W})^{-1} \right\}^n \mathbf{C}_b^\dagger$$

Matching procedure with the standard pert. expansion yields the expressions for $\mathbf{C}_a$, $\mathbf{P}$ and $\mathbf{W}$

$$\mathbf{\Sigma}^{(\text{dyn})\,+}_{ab}(\omega) \equiv \mathbf{\Sigma}^{(\text{dyn},1)\,+}_{ab} + \mathbf{\Sigma}^{(\text{dyn},2)\,+}_{ab} + \dots$$

GOLDSTONE DIAGRAMS

for the polarization propagator

The ADC splits the problem of determing **T** into two tasks: the *construction* of the modified transition ampl. **F** and the *diagonalization* proc. for the modified. interaction matrix, **C** + **K**

- In the ADC for the polarization propag. in energy repres. time integrations are disentangled, by considering the *m+n+2!* possible orderings of the time vertices at order *l=n+m*

Time-ordered or *Goldstone* diagrams are obtained by multiplying each Feynman graph by

$$1 = \theta(t-t') + \theta(t'-t)$$

$$\begin{aligned} 1 &= \theta(t-t')\theta(t_1-t) + \theta(t-t')\theta(t'-t_1) + \theta(t-t_1)\theta(t_1-t') \\ &+ \theta(t'-t)\theta(t_1-t') + \theta(t'-t)\theta(t-t_1) + \theta(t'-t_1)\theta(t_1-t) \\ 1 &= \theta(t'-t_1)\theta(t-t')\theta(t_2-t) + \theta(t_2-t_1)\theta(t'-t_2)\theta(t-t') \\ &+ \theta(t'-t_1)\theta(t_2-t')\theta(t-t_2) + \theta(t_1-t_2)\theta(t'-t_1)\theta(t-t') \\ &+ \theta(t_1-t)\theta(t-t')\theta(t'-t_2) + \ldots \end{aligned}$$

- In practice: each Feynman diagram in $\Pi^{+\,g_1 g_3 g_4 g_1}_{acdb}(t,t')$ corresponds to:

1 Goldstone graph at leading order
3 Goldstone graphs at first order
12 Goldstone graphs at second order
60 Goldstone graphs at third order
...

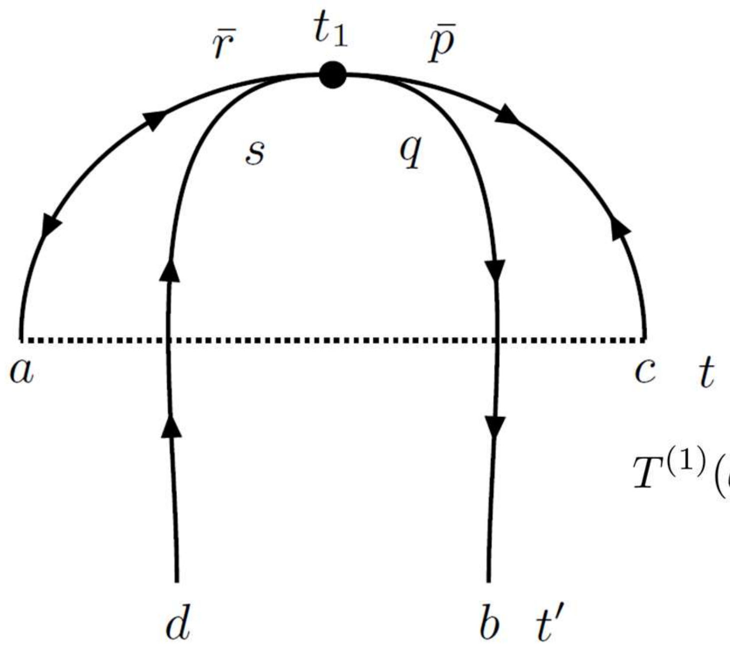
Diagrammatic rules for the Goldstone graphs of the SCGF polarization prop. in energy repr. exist...

⇝ J. Schirmer, *Phys. Rev. A* ***26***, 5, 2395-2416 (1982)

- Example of a *first-order* diagram contributing to $\Pi^{+\,1111}_{acdb}(\omega)$ (conventionally $t > t'$):

in time representation:

$$\Pi^{+\,1111}_{acdb}(t,t') = \ldots + \frac{1}{\hbar}\sum_{pqrs} \bar{v}_{\bar{p}q\bar{r}s} \int_{-\infty}^{+\infty} \mathrm{d}t_1\, G^{(0)\,21}_{pc}(t_1,t^+) G^{11}_{(0)\,bq}(t',t_1) \cdot G^{(0)\,11}_{sd}(t_1,t'^+) G^{(0)\,12}_{ar}(t,t_1)\theta(t-t')\theta(t_1-t) + \ldots$$



Time ordering:

$$t_1 > t > t'$$

- Performing the FT, this Goldstone graph translates into the following contribution to the first order transition function:

$$T^{(1)}(\omega) = \ldots + \sum_{abcd}\sum_{pqrs}\sum_{\substack{k_1\,k_2\\k_3\,k_4}} D^{*}_{ac}\bar{v}_{\bar{p}q\bar{r}s} \frac{{}^{k_1}\chi^{(0)\,2}_{p}\,{}^{k_1}\Upsilon^{(0)\,1}_{c}\,{}^{k_4}\chi^{(0)\,2}_{r}\,{}^{k_4}\Upsilon^{(0)\,1}_{a}}{\omega_{k_1,0}+\omega_{k_2,0}+\omega_{k_3,0}+\omega_{k_4,0}} \cdot \frac{{}^{k_2}\chi^{(0)\,1}_{a}\,{}^{k_2}\Upsilon^{(0)\,1}_{b}\,{}^{k_3}\chi^{(0)\,1}_{s}\,{}^{k_3}\Upsilon^{(0)\,1}_{d}}{\omega-\omega_{k_3,0}-\omega_{k_2,0}+i\eta} D_{db} + \ldots$$

- Due to the SB in Ψ_0, the connection of the 'energies' in the denominators

$$\omega_{k_m0} \equiv \omega_{k_m} - \omega_0$$

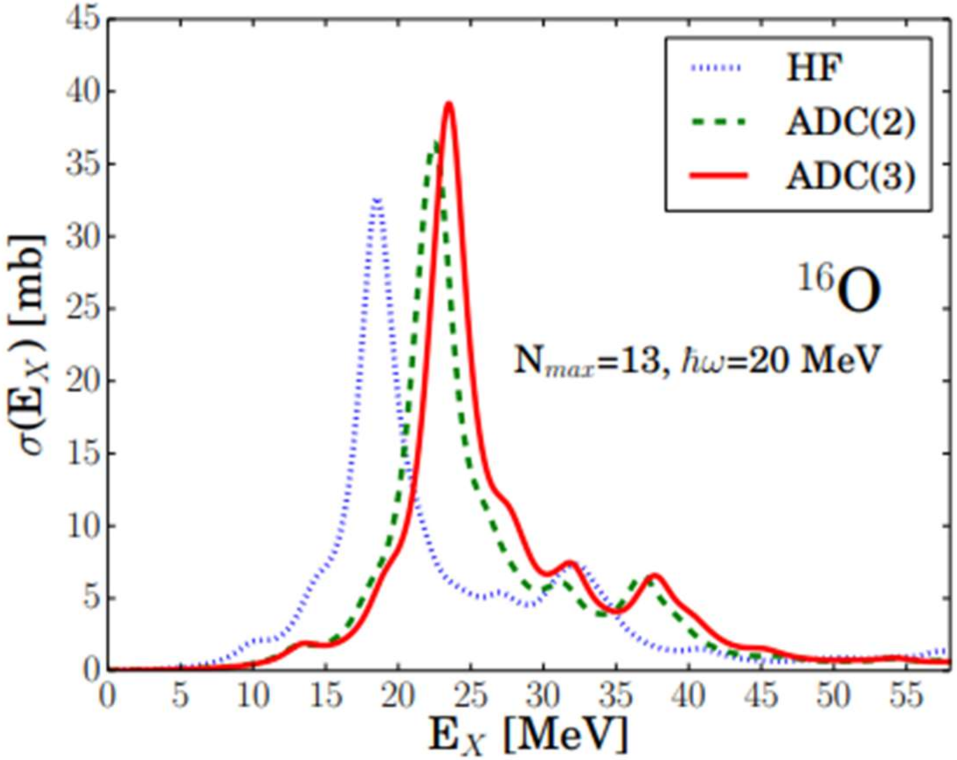
with the single-particle excitation energies is *less transparent*:

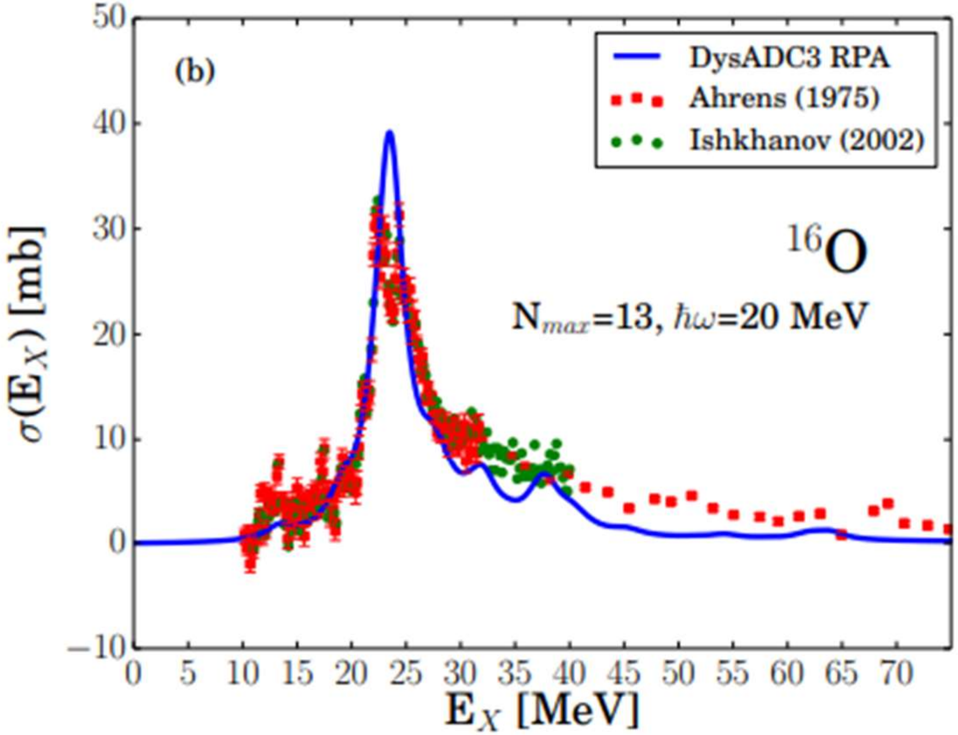
$\omega_{k_1}, \omega_{k_2}, \omega_{k_3}, \omega_{k_4} \Longrightarrow$ Eigenvalues of Ω for states with an odd number of nucleons on average

$\rightsquigarrow$ largest exp. contrib. $= A \pm 1$ states

In SCGF theory, dressed RPA including some 2p-2h excitations is adopted for EM properties of semimagic nuclei with Z = 8, 20, 28 in Dyson GF theory.

Giant dipole resonances are studied, with different parameters of the HO s.p. basis and different implementations of the ADC scheme





[*Phys. Rev. C* ***99***, 054327 (2019)]

The Polarization Propagator in the SC Approach

- Derivation of a *self-consistent equation* for the Gorkov polarization propagator in momentum space.

W. Czyz, *Acta. Phys. Pol.* **20**, 737 (1961).

- Possible *approximation* of the SC equation.

F. Raimondi et al., *Phys. Rev. C* **99**, 054327 (2019).

- Possible *automatisation* of the construction of the necessary Feynman/Goldstone diagrams (ADG).

- Application of the *algebraic diagrammatic construction* scheme to the interaction Kernel

- Implementation of the *angular momentum coupling* (AMC) scheme

A. Tichai et al., *Eur. Phys. J. A* **56**, 272 (2020).

- Redrafting of the *BcDor codes* to include Π

- Application to **semi-magic nuclei**

t

The Seastar Collaboration

Publication of exp. results concerning nuclear spectroscopy campaigns in the period 2014-2017:

Around Z = 20

- ^{47}Cl and ^{49}Cl: *Phys. Rev. C* ***104***, *044331* (2021).
- ^{50}Ar *Phys. Rev. C* ***102***, 064320 (2020), ^{51}Ar *Phys. Lett. B* ***814***, *136108* (2021) and ^{52}Ar *Phys. Rev. Lett.* ***122***, 074502 (2019).
- ^{51}K, ^{53}K *Phys. Lett. B* ***802***, 135215 (2020) and ^{55}K *Phys. Lett. B* ***827,*** 136953 (2022).
- ^{54}Ca *Phys. Rev. Lett.* ***126***, 252501 (2019), ^{55}Ca and ^{57}Ca *Phys. Lett. B* ***827,*** 136953 (2022).
- ^{62}Ti *Phys. Lett. B* ***800,*** 135071 (2020).
- ^{63}V *Phys. Rev. C* ***827,*** 064308 (2021).

Around Z = 28

- ^{72}Fe *Phys. Rev. Lett.* ***115,*** 192501 (2015).
- ^{66}Cr *Phys. Rev. Lett.* ***115,*** 192501 (2015).
- ^{76}Ni *Phys. Rev. C* ***99,*** 014312 (2019) and ^{78}Ni *Nature (London)* ***569,*** 53 (2019).
- ^{79}Cu *Phys. Rev. Lett.* ***119,*** 192501 (2017).
- ^{67}Mn *Phys. Lett. B* ***784,*** 392 (2018).
- ^{84}Zn *Phys. Lett. B* ***773,*** 492 (2017).
- ^{69}Co, ^{71}Co and ^{73}Co *Phys. Rev. C* ***101,*** 034314 (2020).

Legend: one-body propagator; one-body+polarization propagator; not yet investigated;