2. Neutrino Oscillations

2.1 QM of ν oscillations

$$\mathcal{L} = \sum_{j=1,2,3} \Big[\bar{\nu}_j i\not{\partial} \nu_j + \sum_\alpha \frac{g}{\sqrt{2}} \big(\bar{\nu}_{jL} \underbrace{U^*_{\alpha j}}_{\text{mixing matrix}} \gamma^\mu e_{\alpha L} W^\mu + h.c. \big) \Big]$$

$$- \sum_j \left(m_j \bar{\nu}_{jL} \nu_{Rj} + h.c. \right)$$

with $\underbrace{|\nu_\alpha\rangle}_{\text{flavor state}} = \sum_j U^*_{\alpha j} \underbrace{|\nu_j\rangle}_{\text{mass state}}$

Wave function after distance L, time T:

$$|\nu_\alpha(T,L)\rangle = \sum_j U^*_{\alpha j} e^{-iE_j T + ip_j L} |\nu_j\rangle$$

Neutrino detection = projection onto

$$\langle \nu_\beta| = \sum_k U_{\beta k} \langle \nu_k|$$

$\Rightarrow$ Oscillation amplitude

$$\langle \nu_\beta | \nu_\alpha (T, L) \rangle = \sum_j U_{\alpha j}^* U_{\beta j} e^{-iE_j T + i p_j L}$$

Oscillation probability

$$P_{\alpha\beta}(T, L) = |\langle \nu_\beta | \nu_\alpha (T, L) \rangle|^2$$

$$= \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \cdot e^{-i(E_j - E_k)T + i(p_j - p_k)L}$$

In practice: uncertainty in T $\gg$ uncertainty in energy

$\Rightarrow$ integrate over T:

$$P_{\alpha\beta}(L) = \frac{1}{N} \int_{-\infty}^{\infty} dT \; P_{\alpha\beta}(T, L)$$

$$= \frac{1}{N} \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \; \delta(E_j - E_k) \cdot \exp\left[i\left(\sqrt{E^2 - m_j^2} - \sqrt{E^2 - m_k^2}\right) L \right]$$

$|m_j - m_k| \ll E$

$$\simeq \sum_{j,k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \exp\left[-\frac{i\Delta m^2_{jk} L}{2E}\right]$$

with $\Delta m^2_{jk} \equiv m_j^2 - m_k^2$

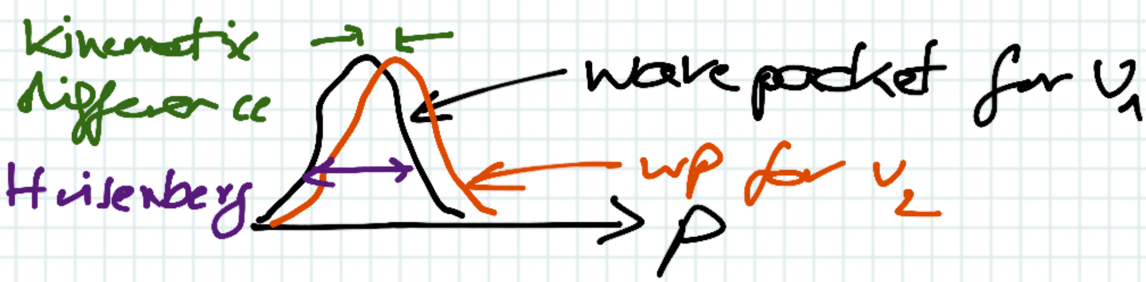
Two-flavor Limit (e, μ)

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\hookrightarrow \boxed{P^{2f}_{e\mu}(L) = \sin^2 2\theta \, \sin^2 \frac{\Delta m^2 L}{4E}}$$

Comment:

States with different E, p can interfere only if the Heisenberg uncertainties $\Delta E, \Delta p$ are larger than the kinematic differences $E_j - E_k$, $p_j - p_k$



2.2 3-flavor oscillations

\# of parameters in U:

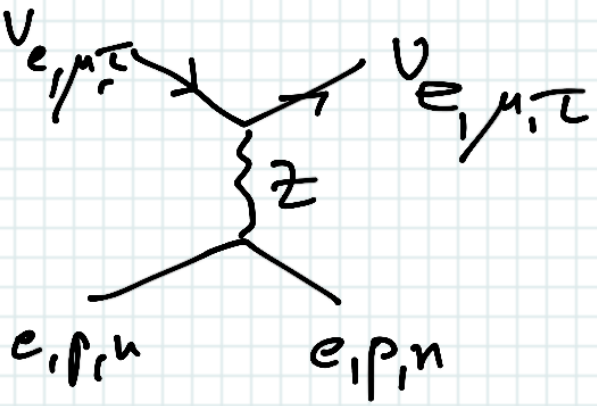
General 3x3:	9 real	9 phases
Unitarity		
$\sum_j \lvert U_{\alpha j}\rvert^2 = 1$	-3	
$\sum_j U_{\alpha j} U^*_{\beta j} = 0$	-3	-3
Field redefinitions		
$\nu_j \to e^{i\phi_j}\nu_j$ $\nu_\alpha \to e^{i\phi_\alpha}\nu_\alpha$		-5
Σ	3	1

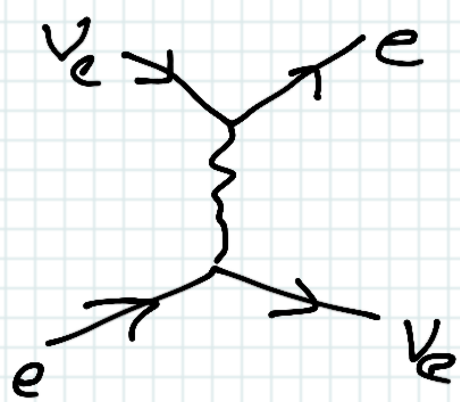
$$\hookrightarrow U = \begin{pmatrix} 1 & & \\ & \overset{\cos\theta_{23}}{c_{23}} & \overset{\sin\theta_{23}}{s_{23}} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}$$

$$\cdot \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

2.3 Oscillation in matter

Weak interactions with e, p, n:





For coherent forward scattering, we cannot tell which e, p, n the ν interacted with $\longrightarrow$ sum over all of them coherently

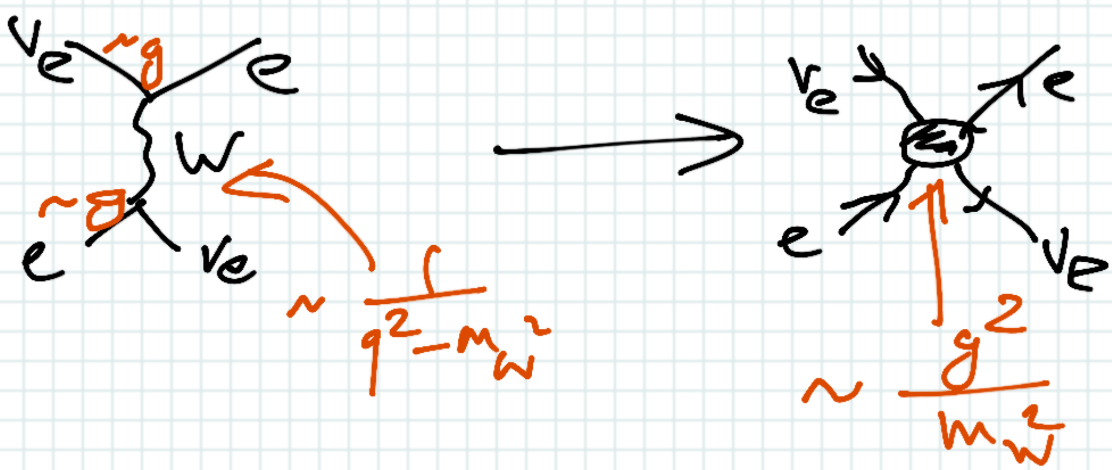
$i\mathcal{M} =$ 

$+ \dots$

Hamiltonian for CC scattering

$$\mathcal{H} \supset \sum_{\alpha} \frac{g}{\sqrt{2}}\left(\bar{\nu}_{\alpha L}\gamma^{\mu} e_{\alpha L} W_{\mu} + h.c.\right)$$

For $E \ll m_W$



$$\Rightarrow \mathcal{H}_{eff} \supset \frac{g^2}{2m_W^2}\left[\bar{\nu}_{eL}\gamma^{\mu} e_L\right]\left[\bar{e}_L \gamma_{\mu} \nu_{eL}\right]$$

$$= \frac{G_F}{\sqrt{2}}\left[\bar{e}\gamma^{\mu}(1-\gamma^5)e\right] \cdot \left[\bar{\nu}_e \gamma_{\mu}(1-\gamma^5)\nu_e\right]$$

Fierz identity

Consider e-field as static

$$\to \langle \bar{e}\gamma^{\mu}(1-\gamma^5)e \rangle$$

$$e = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(u^s(p) e^{-ipx} a_p^s + v^s(p) e^{ipx} b_p^{s\dagger} \right)$$

Assume background e^- have distribution $f(\vec{p})$

$\hookrightarrow \langle \bar{e} \gamma^\mu (1-\gamma^5) e \rangle$

$$= \underbrace{\frac{1}{2}}_{\text{spin average}} \sum_s \int d^3p\, f(\vec{p}) \langle e(\vec{p}, s) | \bar{e} \gamma^\mu (1-\gamma^5) e | \cdot \underbrace{| e(\vec{p}, s) \rangle}_{= a_p^{s\dagger}|0\rangle}$$

$$= \frac{1}{2} \sum_s \int \frac{d^3p}{2E_p} f(p)\, \bar{u}^s(p) \gamma^\mu (1-\gamma^5) u^s(p)$$

$$= \frac{1}{2} \int \frac{d^3p}{2E_p} f(p)\, \underbrace{\mathrm{tr}\left[(\not{p} + m_e) \gamma^\mu (1-\gamma^5) \right]}_{= \mathrm{tr}\, \not{p} \gamma^\mu = 4p^\mu}$$

$$= \begin{cases} n_e & \text{for } \mu = 0 \\ 0 & \text{otherwise} \end{cases}$$

(using $\int d^3p\, f(\vec{p}) = \underbrace{n_e}_{\text{number density of } e^-}$)

$$\Longrightarrow \mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} n_e \left[\bar{\nu}_e \gamma^0 (1-\gamma^5) \nu_e \right]$$

$$= \underbrace{\sqrt{2}\, G_F n_e}_{V_{CC}} \bar{\nu}_{eL} \gamma^0 \nu_{eL}$$

("CC MSW potential")

Impact on oscillations:

$$P_{\alpha\beta}(L) = \sum_{j,k} U^*_{\alpha j} U_{\alpha k} U_{\beta j} U^*_{\beta k} \cdot \exp\left[i(\underbrace{\phi_j}_{= p_j L} - \phi_k) L \right]$$

With V_{MSW}:

$$\hat{p} = \sqrt{(\hat{H} - \hat{V})^2 - \hat{M}^2}$$

$$\simeq \hat{H} - \frac{\hat{M}^2}{2\hat{H}} - \hat{V}$$

in flavor basis:

$$\hat{V} = \begin{pmatrix} V_{cc} & \\ & 0 \end{pmatrix} \quad \text{for 2 flavors}$$

$$\hat{M} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$\sin\theta$ $\quad$ $\cos\theta$

Comment on ν osc. in inhomogeneous matter:

instead of $|\nu_j\rangle \longrightarrow |\nu_j(L)\rangle = e^{ip_jL}|\nu_j\rangle$

we need to solve

$$-i\frac{d}{dx}|\nu_j(x)\rangle = p_j(x)\,|\nu_j(x)\rangle$$