1. **Neutrino oscillations in matter**

(a) Diagonalize the 2-flavor neutrino momentum operator in matter,

$$\hat{p}_{\text{eff}} = -\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\frac{\Delta m^2}{4E} & 0 \\ 0 & \frac{\Delta m^2}{4E} & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} - \begin{pmatrix} \sqrt{2}G_F n_e & 0 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

to show that the effective mass squared difference and mixing angle in matter are given by

$$\frac{\Delta m^2_{\text{eff}}}{2E} = \sqrt{\Big(\sqrt{2}G_F n_e - \frac{\Delta m^2}{2E}\cos 2\theta\Big)^2 + \Big(\frac{\Delta m^2}{2E}\Big)^2 \sin^2 2\theta} \tag{2}$$

$$\sin 2\theta_{\text{eff}} = \frac{\sin 2\theta\, \frac{\Delta m^2}{2E}}{\sqrt{\big(\sqrt{2}G_F n_e - \frac{\Delta m^2}{2E}\cos 2\theta\big)^2 + \big(\frac{\Delta m^2}{2E}\big)^2 \sin^2 2\theta}} \tag{3}$$

(b) Consider now neutrino oscillations in spatially *inhomogeneous* matter of density $n_e = n_e(x)$. This is relevant for instance for solar neutrinos propagating out of the core of the Sun. Neutrino evolution is described by the Schrödinger-like equation

$$-i\frac{d}{dx}\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{\text{eff}} & \sin\theta_{\text{eff}} \\ -\sin\theta_{\text{eff}} & \cos\theta_{\text{eff}} \end{pmatrix} \begin{pmatrix} -\frac{\Delta m^2_{\text{eff}}}{4E} & 0 \\ 0 & \frac{\Delta m^2_{\text{eff}}}{4E} & 0 \end{pmatrix} \begin{pmatrix} \cos\theta_{\text{eff}} & -\sin\theta_{\text{eff}} \\ \sin\theta_{\text{eff}} & \cos\theta_{\text{eff}} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \tag{4}$$

Rewrite this equation in the basis of *matter eigenstates*, i.e. states of definite energy and momentum in matter: $(\nu_A, \nu_B) = U^\dagger_{\text{eff}}(x)\,(\nu_e, \nu_\mu)$, where U_{eff} is the effective mixing matrix in matter. You should find

$$-i\frac{d}{dx}\begin{pmatrix} \nu_A \\ \nu_B \end{pmatrix} = \begin{pmatrix} p_A & i\frac{d\theta}{dx} \\ -i\frac{d\theta}{dx} & p_B \end{pmatrix} \begin{pmatrix} \nu_A \\ \nu_B \end{pmatrix}, \tag{5}$$

where p_A, p_B are the momentum eigenvalues in matter.

(c) For slowly varying matter density, $d\theta/dx \ll |p_A - p_B|$, the off-diagonal terms on the right hand side of eq. (5) are neglgible. Solve the equation to show that the survival probability of solar neutrinos $P(\nu_e \to \nu_e)$ is given by

$$P(\nu_e \to \nu_e) = \frac{1}{2}\bigg(1 + \cos 2\theta_i \cos 2\theta_f + \sin 2\theta_i \sin 2\theta_f \cos\bigg[\int_{x_i}^{x_f} dx\, \frac{\Delta m^2_{\text{eff}}(x)}{2E}\bigg]\bigg). \tag{6}$$

Here θ_i and θ_f are the effective mixing angles corresponding to the center of the Sun and its exterior, respectively. The integral in the last term runs along the neutrino trajectory from its production point x_i to its detection point x_f.

(d) What is the maximum conversion probability in the case of small vacuum mixing angle $\theta \ll 1$? Consider the case that the matter density at the center of the Sun lies well above the MSW resonance, whereas the density outside the Sun is far below.

2. Imagine a world in which neutrinos are massive, but charged leptons are massless. Will neutrinos oscillate in such a world?

3. Do neutrinos produced in the decay $Z^0 \to \bar{\nu}\nu$ oscillate? If so, describe a gedanken-experiment in which these oscillations could be observed.

4. **Atmospheric neutrinos**
   Explain why the initial flavor ratio of atmospheric neutrinos at energies $\lesssim$ few GeV is $\overset{(-)}{\nu}_e : \overset{(-)}{\nu}_\mu : \overset{(-)}{\nu}_\tau \sim 1 : 2 : 0$. Consider the main products of the interactions of high-energy cosmic ray protons with nuclei in the Earth's atmosphere.

5. Discuss the advantages and disadvantages of a hypothetical neutrino oscillation experiment using a mono-energetic beam of $\mathcal{O}(\text{GeV})$ neutrinos. Can you think of a possible realization of such an experiment?

6. Imagine there is a fourth ("sterile") neutrino flavor in nature, with $U_{e4} \simeq U_{\mu 4} \simeq 0.1$, and with the corresponding fourth mass eigenvalue being of order $m_4 \sim 1$ eV. Discuss the phenomenological consequences. Which approximations can be made in computing oscillation probabilities between the fourth neutrino and the three standard ones? How would you test this hypothesis experimentally? Describe at least two possible experimental setups.