Muon Cascade Calculations

Philip D. Jones

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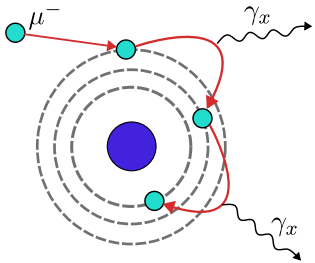

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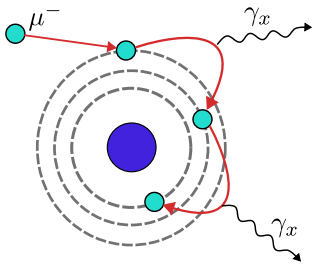

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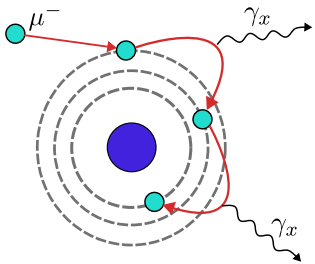

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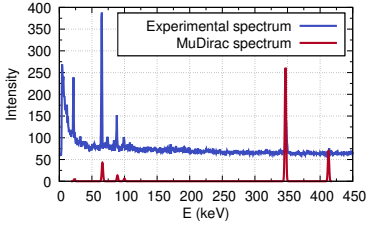

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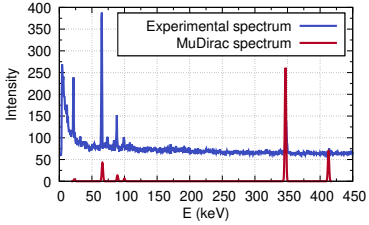
- Energies are currently predicted well when compared to experiment, but not intensities



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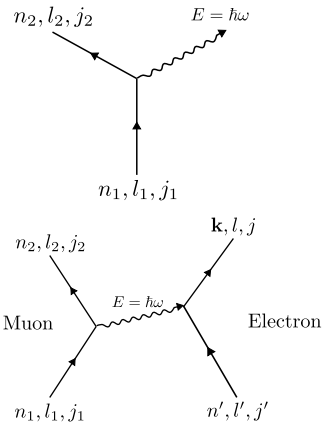

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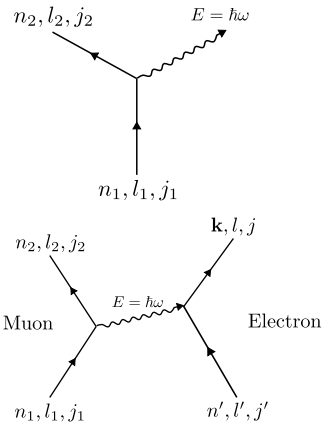

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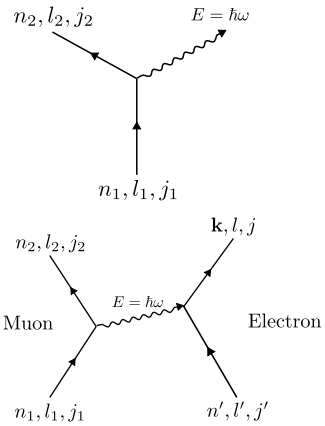

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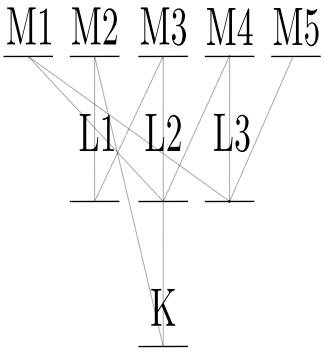

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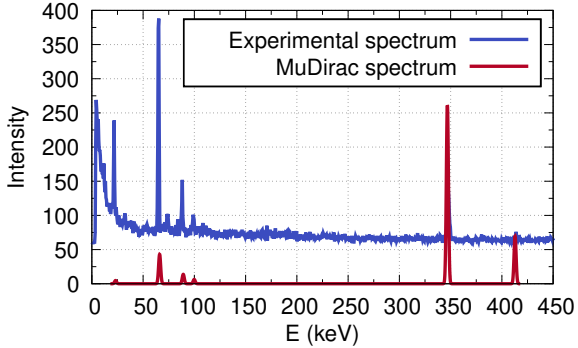
- The final measurable intensities are therefore a function of all possible previous transitions

X-ray Intensities

- Currently `MuDirac`[1], a software for calculating muonic x-ray emission spectra, doesn't get the intensities right



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- This is useful for validation of new relativistic results

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- Energies are calculated accurately through the use of Dirac spinors, finite nuclear models, and QED corrections
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- Radiative dipoles are calculated, but Auger transitions, electron refilling, and angular momentum distributions are not considered

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Relativistic Radiative Transitions

- The Dirac spinors are coupled to the electromagnetic field with an interaction Hamiltonian

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$$\Gamma_R = \frac{4E_{if}}{3c} |\langle \psi_f | (\boldsymbol{\alpha} \cdot \hat{\mathbf{e}}_\lambda) e^{-i\mathbf{k}_\lambda \cdot \mathbf{r}} | \psi_i \rangle|^2 \tag{2}$$

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- The exponential can be expanded out in Taylor series, and each term in the expansion corresponds to a term in the multipole expansion

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- Comparison to the Akylas-Vogel code is valid for systems in the non-relativistic limit i.e low Z atoms
- Individual quadrupole rates were compared for a given set of transitions:

(n,l) pairs	**Akylas-Vogel**	**Fine pairs**	**MuDirac**
(3,2)→(1,0)	5.48×10^9	K1-M4	3.70×10^9
		K1-M5	1.61×10^9
		Sum	5.31×10^9
(3,2)→(2,0)	4.72×10^8	L1-M4	3.18×10^8
		L1-M5	1.39×10^8
		Sum	4.57×10^8

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Auger Transitions

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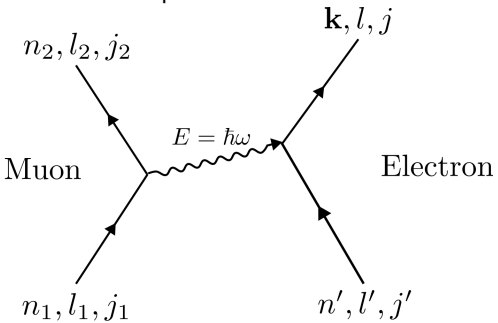

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- Four wavefunctions are needed to calculate this: the initial muon state, final muon state, a bound electron, and a continuum electron

$$\Gamma_A = \left| \frac{e^2}{\hbar} \int \int \chi_f{}^*(\mathbf{r}_2) \psi_f{}^\dagger(\mathbf{r}_1) \frac{1}{||\mathbf{r}_1 - \mathbf{r}_2||} \chi_i(\mathbf{r}_2) \psi_i(\mathbf{r}_1) d^3\mathbf{r}_1 d^3\mathbf{r}_2 \right|^2$$

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- Coulomb operator can be expanded out in spherical harmonics using the Laplace expansion, and expressions for Auger multipole rates were obtained
- Validation against Akylas-Vogel is yet to be performed

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- TCDFT has been used for electron-positive muon systems, including a parameterisation of the correlation functional
- Being able to solve this problem for the negative muon would give a more accurate description of the system

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- A natural route to the full interacting muon-electron problem would come through this implementation
- Ultimately, the ability to solve for muonic states in a bulk system may allow for more accurate capture probabilities, potentially leading to better intensities

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- `MuDirac` does not get intensities correct due to missing physics
- Relativistic radiative and Auger transitions are the first steps taken to getting correct intensities, with the radiative rates being successfully validated against the Akylas-Vogel code

References

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