An index for interacting topological phases

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Classifying quantum phases

Goal: Classify phases of matter at $T = 0$

- Landau: use a local order parameter
- Not complete (Wen-Niu, AKLT, Kitaev, Levin-Wen): local disorder but topological order
- Replace the local order parameter by a stable discrete index

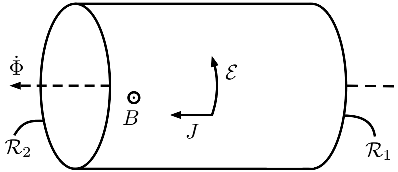
E.g.: Hall conductance $(\mathbb{Z}, \mathbb{Z}/q)$, indices of topological insulators $(\mathbb{Z}, \mathbb{Z}_2)$

This talk: An index associated to ground states and a $U(1)$-charge

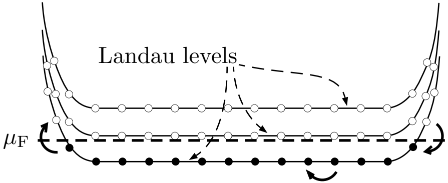
- for interacting electrons
- taking rational values in $\mathbb{Z}/q$
- where q is the topological degeneracy

Motivation: Laughlin's argument

The Laughlin pump:



Adiabatic increase of Φ:



The Hall conductance is an index (in the punctured plane geometry):

$$\begin{aligned} 2\pi\sigma_{\mathrm{H}} &= \mathrm{Ind}(P, UPU^*) \\ &= \mathrm{Tr}((P - UPU^*)^3) \\ &= \mathrm{dimKer}(P - UPU^* - 1) - \mathrm{dimKer}(P - UPU^* + 1) \in \mathbb{Z} \end{aligned}$$

where P is the Fermi projection and U adds a unit of flux

Charge transport across a line

Kitaev's flow of a unitary

$$\mathcal{F}(U) = \sum_{j\leq 0, k>0} \left(|U_{jk}|^2 - |U_{kj}|^2\right)$$

Example, translation: $$U = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & \ddots & \ddots \end{pmatrix} \implies \mathcal{F}(U) = 1$$

Local index

Interpretation: U transports 1 charge across the fiducial line $j = 0$

Formal computation:

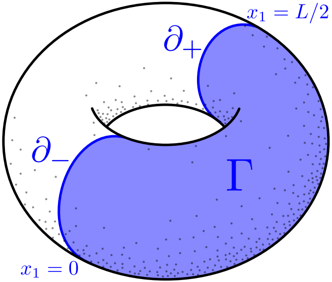
$$\mathcal{F}(U) = \mathrm{Tr}(U^* Q U(1-Q)) - \mathrm{Tr}(U^*(1-Q)UQ) = \mathrm{Tr}(U^*QU - Q)$$

Quantum lattice system

- Charge at site x is $q_x = a_x^* a_x$ and the charge on a half space

$$Q = \sum_{1 \le x_1 < L/2} q_x$$

- Unitary operator U transporting charge: translation, flux insertion,...



- Continuity equation: The charge transport operator

$$T = U^* Q U - Q$$

is supported around $\partial_- \cup \partial_+$, and

$$T = T_- + T_+ + \mathcal{O}(L^{-\infty})$$

Ground states

P is a ground state projection

- of a charge conserving Hamiltonian

$$\mathrm{i}[H, Q_Z] \text{ supported in } \partial Z$$

- having a gap above the ground state energy
- with topological order

$$q = \mathrm{Rank}(P)$$

and any local observable A acts trivially in the ground state space:

$$\|PAP - c(A)P\| = \mathcal{O}(L^{-\infty})$$

Invariance under U:

$$[U, P] = \mathcal{O}(L^{-\infty})$$

(translation invariance, insertion of a unit of flux,...)

The index

Theorem. [B.-Bols-De Roeck-Fraas]
Assume Q, U, P as above. For any ground state $\Omega = P\Omega$, Then,

$$\mathrm{dist}\big(q\langle\Omega|T_-|\Omega\rangle, \mathbb{Z}\big) = \mathcal{O}(L^{-\infty})$$

- Recall: $U^*QU - Q = T_- + T_+ + \mathcal{O}(L^{-\infty})$
  q is the degeneracy
- If the limit exists, define

$$\mathrm{Ind}(U,\Omega) := \lim_{L\to\infty} \langle\Omega, T_-\Omega\rangle \in \frac{\mathbb{Z}}{q}$$

- Stable under perturbations of U
- Stable under perturbations of P that keep the gap open

Various topological indices

Physical realizations: Choose U

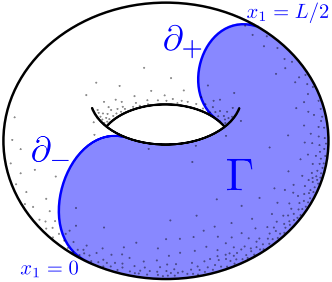
- Adding flux (Laughlin pump): fractional Hall conductance
- Translation: Lieb-Schultz-Mattis theorem, fractional filling
- Adiabatic evolution along a cycle: Thouless pump, fractional charge transport
- Propagator $U = \exp(\mathrm{i}tH)$: Bloch's theorem, vanishing currents

All of that in an interacting setting, assuming a gap

Local charge fluctuations

Useful fact:
One can construct $K_\pm$ localized near $\partial_\pm$ such that

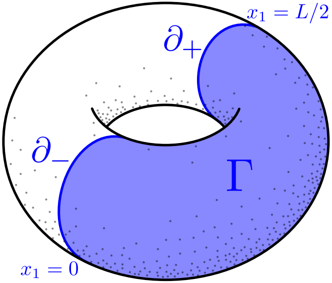
$$[Q - K_- - K_+, P] = \mathcal{O}(L^{-\infty})$$



- ▷ $\overline{Q} := Q - K_- - K_+$ leaves the ground state space invariant
- ▷ $Q \to \overline{Q}$ affects only fluctuations:

$$\mathrm{Tr}(P(U^*\overline{Q}U - \overline{Q})_-) = \mathrm{Tr}(P(U^*QU - Q)_-)$$

Local charge fluctuations



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How?

$$K_- + K_+ = \int_{-\infty}^{\infty} W(t)\mathrm{e}^{\mathrm{i}tH_s}\mathrm{i}[H,Q]\mathrm{e}^{-\mathrm{i}tH_s}dt$$

Full counting statistics

The operator

$$Z(\lambda) = U^* \mathrm{e}^{\mathrm{i}\lambda\overline{Q}} U \mathrm{e}^{-\mathrm{i}\lambda\overline{Q}}$$

- acts on the range of P
- factorizes

$$Z(\lambda) \simeq Z_-(\lambda) Z_+(\lambda) \qquad Z_-(\lambda) = \mathrm{e}^{\mathrm{i}\lambda\overline{Q}_-^U} \mathrm{e}^{-\mathrm{i}\lambda\overline{Q}_-}$$

Key actor:

$$\chi(\lambda) = \det(P Z_-(\lambda) P)$$

describes the statistics of charge transport across ∂_-

A winding number

We claim that

$$-\mathrm{i}\chi'(\lambda) \simeq \mathrm{Tr}(PT_-)\chi(\lambda)$$

Follows from

$$\frac{d}{d\lambda}\det(A(\lambda)) = \mathrm{Tr}(A(\lambda)^{-1}A'(\lambda))\det(A(\lambda))$$

and some algebra

Hence

$$\chi(\lambda) \simeq \mathrm{e}^{\mathrm{i}\lambda q\langle T_-\rangle_P}$$

and it suffices to show

$$\chi(2\pi) \simeq 1$$

to prove the theorem

Remark on braiding

With the assumption of topological order:

$$PT_{-}P \simeq \frac{n}{q}P$$

so we actually showed

$$\mathcal{U}^*\mathcal{V}^*\mathcal{U}\mathcal{V} \simeq \mathrm{e}^{2\pi\mathrm{i}\frac{n}{q}} \qquad (\mathcal{U} = PUP,\ \mathcal{V} = P\mathrm{e}^{2\pi\mathrm{i}\overline{Q}_{-}}P)$$

as an equality between unitary matrices on the ground state space

- Braiding relation
- Irreducible representation is q-dimensional: fractional Hall conductance related to topological ground state degeneracy
(see also Wen-Niu 1990)

Concluding remarks

- ▷ Fractional charge transport in interacting setting
- ▷ Combining translation and flux increase:
  constraint between Hall conductance and filling factor
- ▷ Generalizes to the case of discrete local symmetry breaking
- ▷ Topological quantum numbers without topology