HVP Contributions to Muon g-2: QCD Constraints Using Inequalities and Sum Rules

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Muon Anomalous Magnetic Moment a_μ

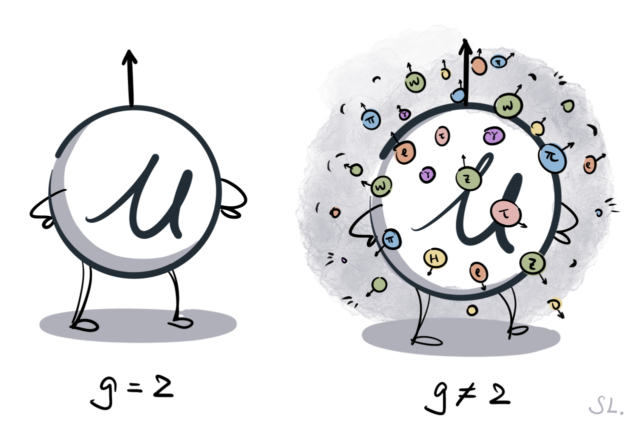


Illustration by Siyuan Li *(inspired by "The Muon g–2 Anomaly Explained" from aps.org)*

$$a_\mu = (g-2)_\mu / 2$$

Hadronic Vacuum Polarization (HVP) contributions to a_μ

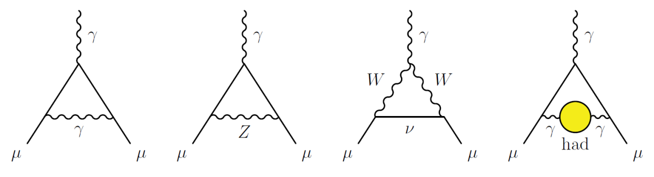


Fig 56.1 *[S. Navas et al. (Particle Data Group), Phys. Rev. D 110, 030001 (2024)]*

$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{EW}} + \mathbf{a}_\mu^{\mathrm{had}}$$

(Aoyama, et al. 2020)

$$a_\mu^{\mathrm{QED}} \rightarrow 116584718.931(104) \times 10^{-11}$$

$$a_\mu^{\mathrm{EW}} \rightarrow 153.6(1.0) \times 10^{-11}$$

$$\mathbf{a}_\mu^{\mathrm{had}} \rightarrow 6931(40) \times 10^{-11}$$

Tension in Theoretical Predictions

For $\mathbf{a}_\mu^{\mathrm{HVP,LO}}$,

- Fermilab Muon $g-2$ Experiment[1] disagrees with theory[2] at 5.0σ
- CMD-3 detector for pion[3]
- Lattice QCD[4] (*e.g.*, BMW collaboration)

...

us:

QCD bounds on leading-order hadronic vacuum polarization contributions to the muon anomalous magnetic moment

Siyuan Li, T.G. Steele, J. Ho, R. Raza, K. Williams, R.T. Kleiv

Phys.Rev.D 110 (2024) 1, 014046, 2404.08591 [hep-ph]

[1] D. P. Aguillard *et al.* [Muon g-2], Phys. Rev. Lett. 131 (2023) no.16, 161802

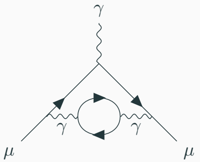
[2] T. Aoyama *et al.* Phys. Rept. 887 (2020), 1-166

[3] F. V. Ignatov *et al.* [CMD-3], Phys. Rev. Lett. 132 (2024) no.23, 231903

[4] S.Borsanyi *et al.* Nature 593 (2021) no.7857, 51-55

Methodology

Finite-energy QCD sum rule (FESR)



Dispersion integral

$$a_\mu^{\mathrm{HVP,LO}} = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty}\sigma^H(t)K(t)\,\mathrm{d}t \approx \frac{4m_\mu^2\alpha^2}{3\pi}\int_{4m_\pi^2}^{\infty}\frac{1}{t^2}\mathrm{Im}\Pi^H(t)\,\mathrm{d}t$$

Finite-energy QCD sum rule (FESR) structure

$$F_k(s_0) = \int_{t_0}^{s_0}\frac{1}{\pi}\mathrm{Im}\Pi^H(t)\,t^k\,\mathrm{d}t\,,$$

$$\Rightarrow \quad a_\mu^{\mathrm{QCD}} \approx \frac{4m_\mu^2\alpha^2}{3}F_{-2}(\infty)$$

Hölder inequality (Lower Bound)

$$\left|\int_{t_1}^{t_2} f\,(t)\,g\,(t)\,\mathrm{d}\mu\right| \leq \left(\int_{t_1}^{t_2} |f\,(t)|^p \mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |g\,(t)|^q \mathrm{d}\mu\right)^{\frac{1}{q}},\ \frac{1}{p}+\frac{1}{q}=1\,.$$

$$\Downarrow$$

$$F_k^2 \leq F_{k+1}F_{k-1}\,.$$

$$\Downarrow$$

$$F_{-2} \geq \frac{F_0^3}{F_1^2}\,.$$

Upper Bound

Method inspired by *F. Dalfovo and S. Stringari, Phys. Rev. B 46, 3991 (1992).*

$$\int_{t_0}^{s_0} \frac{1}{t^2}\left[1+At\right]^2 \mathrm{Im}\Pi^H(t)\,\mathrm{d}t \le \begin{cases} \frac{1}{t_0}\int_{t_0}^{s_0} \frac{1}{t}\left[1+At\right]^2 \mathrm{Im}\Pi^H(t)\,\mathrm{d}t \\ \frac{1}{t_0^2}\int_{t_0}^{s_0} \left[1+At\right]^2 \mathrm{Im}\Pi^H(t)\,\mathrm{d}t \end{cases}$$

eg. $\int_{t_0}^{s_0} \left[\frac{1}{t^2}+\frac{2A}{t}+A^2\right] \mathrm{Im}\Pi^H(t)\,\mathrm{d}t \;\sim\; F_{-2}+2A\,F_{-1}+A^2\,F_0$

$$\xi\frac{4m_\mu^2\alpha^2}{3}\frac{F_0^3(s_0)}{F_1^2(s_0)} \le a_\mu^{\mathrm{QCD}} \le \frac{4m_\mu^2\alpha^2}{3}\begin{cases} F_{-1}^{(B)}/t_0 - \frac{\left(F_0/t_0-F_{-1}^{(B)}\right)^2}{F_1/t_0-F_0} \\ F_0/t_0^2 - \frac{\left(F_1/t_0^2-F_{-1}^{(B)}\right)^2}{F_2/t_0^2-F_0} \end{cases},$$

where $\xi = 0.83\,,\; F_{-1}^{(B)} \equiv \frac{F_0}{t_0} - \frac{(F_1/t_0-F_0)^2}{(F_2/t_0-F_1)}.$

QCD Inputs

The QCD correlation function for the (single) light quark vector current $j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$:

$$\Pi\left(Q^2\right) = \frac{1}{4\pi^2}\Pi^{\mathrm{pert}}\left(Q^2\right) - \frac{3m_q^2(\nu)}{2\pi^2 Q^2} + 2\langle m_q \bar{q}q\rangle \frac{1}{Q^4}\left(1 + \frac{1}{3}\frac{\alpha_s(\nu)}{\pi}\right)$$
$$+ \frac{1}{12\pi}\langle \alpha_s G^2\rangle \frac{1}{Q^4}\left(1 + \frac{7}{6}\frac{\alpha_s(\nu)}{\pi}\right) - \frac{224}{81}\pi\alpha_s\langle \bar{q}\bar{q}qq\rangle \frac{1}{Q^6}$$
$$+ \cdots$$

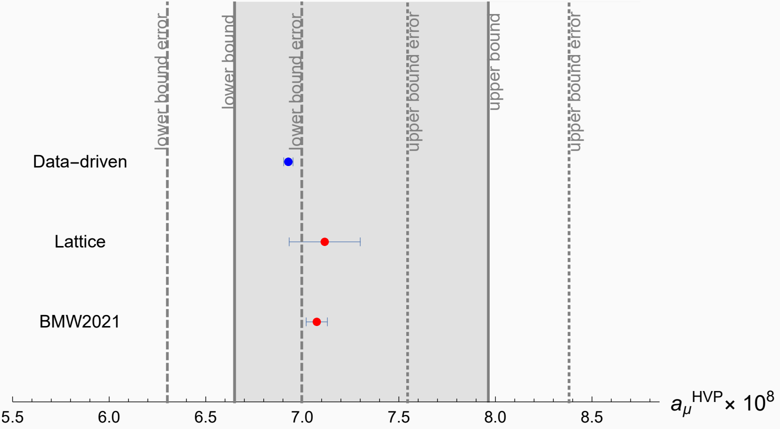
QCD Inputs (FESR)

$$F_0\left(s_0\right) = \frac{1}{4\pi^2}\left[1 + \frac{\alpha_s(\nu)}{\pi}T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi}\right)^2\left(T_{2,0} + T_{2,1}\right) + \left(\frac{\alpha_s(\nu)}{\pi}\right)^3\left(T_{3,0} + T_{3,1} + 2T_{3,2}\right)\right.$$
$$\left. + \left(\frac{\alpha_s(\nu)}{\pi}\right)^4\left(T_{4,0} + T_{4,1} + 2T_{4,2} + 6T_{4,3}\right)\right]s_0 - \frac{3}{2\pi^2}m_q(\nu)^2\,,$$

$$F_1\left(s_0\right) = \frac{1}{8\pi^2}\left[1 + \frac{\alpha_s(\nu)}{\pi}T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi}\right)^2\left(T_{2,0} + \frac{1}{2}T_{2,1}\right) + \left(\frac{\alpha_s(\nu)}{\pi}\right)^3\left(T_{3,0} + \frac{1}{2}T_{3,1} + \frac{1}{2}T_{3,2}\right)\right.$$
$$\left. + \left(\frac{\alpha_s(\nu)}{\pi}\right)^4\left(T_{4,0} + \frac{1}{2}T_{4,1} + \frac{1}{2}T_{4,2} + \frac{3}{4}T_{4,3}\right)\right]s_0^2$$
$$- 2\langle m_q\bar{q}q\rangle\left(1 + \frac{1}{3}\frac{\alpha_s(\nu)}{\pi}\right) - \frac{1}{12\pi}\langle\alpha_s G^2\rangle\left(1 + \frac{7}{6}\frac{\alpha_s(\nu)}{\pi}\right)\,,$$

$$F_2\left(s_0\right) = \frac{1}{12\pi^2}\left[1 + \frac{\alpha_s(\nu)}{\pi}T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi}\right)^2\left(T_{2,0} + \frac{1}{3}T_{2,1}\right) + \left(\frac{\alpha_s(\nu)}{\pi}\right)^3\left(T_{3,0} + \frac{1}{3}T_{3,1} + \frac{2}{9}T_{3,2}\right)\right.$$
$$\left. + \left(\frac{\alpha_s(\nu)}{\pi}\right)^4\left(T_{4,0} + \frac{1}{3}T_{4,1} + \frac{2}{9}T_{4,2} + \frac{2}{9}T_{4,3}\right)\right]s_0^3 - \frac{224}{81}\pi\alpha_s\langle\bar{q}\bar{q}qq\rangle\,.$$

Results



*supplemented with charmonium and bottomonium resonance contributions [5] for comparison.

[5] A. Keshavarzi, D. Nomura, and T. Teubner, Phys. Rev. D 101, 014029 (2020).
T. Aoyama et al., Physics Reports 887, 1 (2020).
S. Borsanyi et al., Nature 593, 51 (2021), [arXiv:2002.12347 [hep-lat]].

Conclusion

- Construct constraints on $a_\mu^{\mathrm{HVP,LO}}$ using FESR and Hölder inequalities

$$(657.0 \pm 34.8) \times 10^{-10} \leq a_\mu^{\mathrm{HVP,LO}} \leq (788.4 \pm 41.8) \times 10^{-10}\,;$$

- Bridge the gap between LQCD and data-driven approaches;
- A possible resolution of the tension between LQCD and data-driven determinations of $a_\mu^{\mathrm{HVP,LO}}$.

Publication

QCD bounds on leading-order hadronic vacuum polarization contributions to the muon anomalous magnetic moment

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Phys.Rev.D 110 (2024) 1, 014046, arXiv: 2404.08591v2 [hep-ph]

Thank you! … Questions?

Backup slides: How is a_μ measured?

Measurement of the Positive Muon Anomalous Magnetic Moment to 0.20 ppm
Phys. Rev. Lett. 131, 161802 – Published 17 October 2023
Detailed report on the measurement of the positive muon anomalous magnetic moment to 0.20 ppm
Phys. Rev. D 110, 032009 – Published 8 August 2024

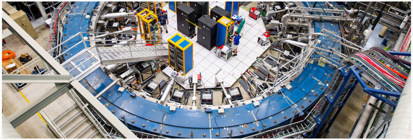


image source: muon-g-2.fnal.gov

pion decay produces muons

$\downarrow$

muon in the storage ring with uniform magnetic field $\vec{B}$

$\downarrow$

precession of muons (g factor)

$\downarrow$

muon decay $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$

$\downarrow$

direct measure of a_μ

$\Rightarrow$ Cyclotron frequency $\omega_c = eB/m$

$\Rightarrow$ torque on muon spin by the magnetic field ($\vec{\mu} \times \vec{B}$) with the muon spin precession frequency

$\omega_s = 2\mu B = g\frac{eB}{2m} = (1 + a_\mu)\frac{eB}{2m}$

$\Rightarrow$ anomalous precession frequency

$\omega_a = \omega_s - \omega_c = a_\mu \frac{e}{m} B$

Backup slides: How is a_μ measured?

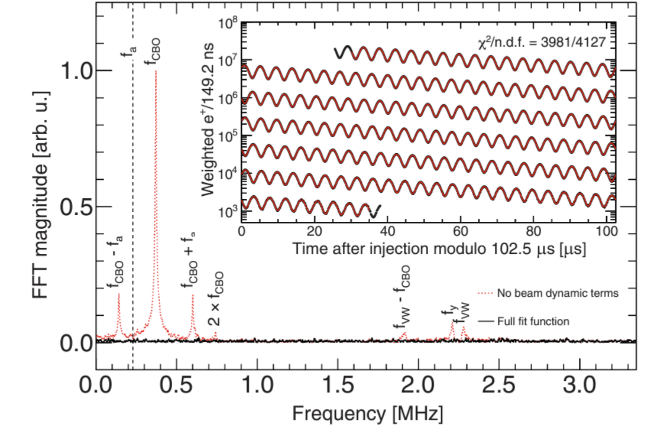


Fig.1 & Eq.(3)
D. P. Aguillard et al. [Muon g-2], Phys. Rev. Lett. 131 (2023) no.16, 161802 [2308.06230 [hep-ex]]

$$N(t) = N_0\, \eta_N(t)\, e^{-t/\gamma\tau_\mu} \times \{1 + A\eta_A(t) \cos[\omega_a t + \varphi_0 + \eta_\phi(t)]\}$$

Backup slides: QCD Sum Rules (QCDSR)

Asymptotic freedom: As the distance between interacting particles decreases, the energy scale increases and the strength of the strong interaction decreases. $\rightarrow$ **colour confinement**

- **Two-point correlation function** $\Pi(q^2)$
- **Operator Product Expansion (OPE)**

$$\Pi(q^2) \sim \sum_n C_n(q^2) \langle \mathcal{O}_n \rangle = C_1(q^2)\mathbf{1} + C_4(q^2) \left\langle G^2 \right\rangle + \cdots ,$$

- **Dispersion Relation (quark-hadron duality)**

$$\Pi(Q^2) = \int_{s_0}^{\infty} ds \, \frac{\rho^{\text{had}}(s)}{s + Q^2},$$

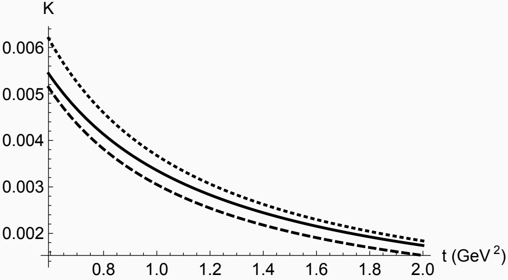
- **Extract hadronic parameters** by equating the correlator OPE (LHS) and the phenomenological representation (RHS)

Backup slides: Dimensional Regularization

Regularize divergence:

- 4-dimensional integral $\rightarrow$ d-dimensional
- Expand in $d = 4 \pm 2\epsilon$ dimensions as $\epsilon \rightarrow 0$
- divergences are in $\mathcal{O}(\frac{1}{\epsilon})$ terms

Backup slides: Kernel Function Approximation



The exact $K(t)$ (solid line) compared to the approximate form $K_\xi(t)$ with $\xi = 0.83$ (lower dashed line) and with $\xi = 1$ (upper dotted line).

[Retrieved from Fig. 1 of arXiv: 2404.08591 [hep-ph]]

$$K(t) \approx \frac{m_\mu^2}{3t} = K_{\mathrm{approx}}(t)$$

Backup Slides: Hölder inequality (Lower Bound)

The general Hölder inequality

$$\left|\int_{t_1}^{t_2} f\,(t)\,g\,(t)\,\mathrm{d}\mu\right| \leq \left(\int_{t_1}^{t_2} |f\,(t)|^p \mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |g\,(t)|^q \mathrm{d}\mu\right)^{\frac{1}{q}},\ \frac{1}{p}+\frac{1}{q}=1\,.$$

With $\mathrm{d}\mu = \frac{1}{\pi}\mathrm{Im}\Pi^H\,(t)\mathrm{d}t$ and careful choice of functions $f(t)$, $g(t)$:

$$F_{\alpha+\beta}\,(s_0) \leq [F_{\alpha\,p}\,(s_0)]^{\frac{1}{p}} \left[F_{\frac{\beta\,p}{p-1}}\,(s_0)\right]^{\frac{p-1}{p}}\ .$$

The most restrictive lower bound (from the Cauchy-Schwarz inequality) :

$$F_{-2} \geq \frac{\mathbf{F_0^3}}{\mathbf{F_1^2}} \geq \frac{F_0^2}{F_2} \geq \frac{F_1^4}{F_2^3}\,.$$

Backup Slides: Our Constraints

Constraint: the Cauchy-Schwarz inequality (*i.e.*, the Hölder inequality with $p = 2$, $\alpha = \frac{k+1}{2}$ and $\beta = \frac{k-1}{2}$)

$$F_k^2 \leq F_{k+1} F_{k-1} \,.$$

$$\xi \frac{4m_\mu^2 \alpha^2}{3} \frac{F_0^3\left(s_0\right)}{F_1^2\left(s_0\right)} \leq a_\mu^{\mathrm{QCD}} \leq \frac{4m_\mu^2 \alpha^2}{3} \begin{cases} F_{-1}^{(B)}/t_0 - \frac{\left(F_0/t_0 - F_{-1}^{(B)}\right)^2}{F_1/t_0 - F_0} \\ \\ F_0/t_0^2 - \frac{\left(F_1/t_0^2 - F_{-1}^{(B)}\right)^2}{F_2/t_0^2 - F_0} \end{cases} ,$$

where $\xi = 0.83\,.$

Backup Slides: QCD Inputs

The QCD correlation function for the (single) light quark vector current $j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$:

$$\begin{aligned}\Pi\left(Q^2\right) =& \frac{1}{4\pi^2}\Pi^{\mathrm{pert}}\left(Q^2\right) - \frac{3m_q^2(\nu)}{2\pi^2 Q^2} + 2\langle m_q\bar{q}q\rangle\frac{1}{Q^4}\left(1+\frac{1}{3}\frac{\alpha_s(\nu)}{\pi}\right)\\ &+\frac{1}{12\pi}\langle\alpha_s G^2\rangle\frac{1}{Q^4}\left(1+\frac{7}{6}\frac{\alpha_s(\nu)}{\pi}\right) - \frac{224}{81}\pi\alpha_s\langle\bar{q}\bar{q}qq\rangle\frac{1}{Q^6}\\ &+\cdots\end{aligned}$$

The perturbative contributions:

$$\frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}\left(t,\nu\right) = 1 + \sum_{n=1}^{\infty}\frac{\alpha_s(\nu)}{\pi}^n\sum_{m=0}^{n-1}T_{n,m}\log^m\left(\frac{\nu^2}{t}\right).$$

Backup slides: Laplace Sum Rule Approach

QCD Laplace sum-rules

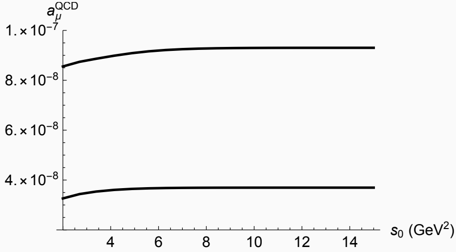
$$L_k(\tau, s_0) = \int_{t_0}^{s_0} \frac{1}{\pi} \mathrm{Im}\Pi^H(t)\, t^k\, e^{-t\tau}\, \mathrm{d}t\,.$$

Approximate Kernel function near $t = t'$:

$$K(t) \approx \mathcal{K}(t, t') = K(t')\, e^{\zeta} \left[a_1 \left(\frac{t}{t'}\right) + a_2 \left(\frac{t}{t'}\right)^2 + a_3 \left(\frac{t}{t'}\right)^3 \right] e^{-\zeta t/t'}\,,$$

$$a_\mu^{\mathrm{QCD}} \approx 4\alpha^2\, K(\zeta/\tau)\, \frac{\tau}{\zeta} e^{\zeta} \left[a_1 L_0(\tau, s_0) + a_2 \frac{\tau}{\zeta} L_1(\tau, s_0) + a_3 \left(\frac{\tau}{\zeta}\right)^2 L_2(\tau, s_0) \right]\,,$$

where $a_1 + a_2 + a_3 = 1$, $t_0 = 4m_\pi^2$, $\tau = \zeta/t'$.



$$369.5 \times 10^{-10} \le a_\mu^{\mathrm{QCD}} \le 930.2 \times 10^{-10}$$

Backup slides: Finding coefficients $T_{n,m}$

$N_f = 4$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$N_f = 3$	$m = 0$	$m = 1$	$m = 2$	$m = 3$
$n = 1$	1	–	–	–	$n = 1$	1	–	–	–
$n = 2$	1.52453	25/12	–	–	$n = 2$	1.63982	9/4	–	–
$n = 3$	-11.6856	9.56054	625/144	–	$n = 3$	-10.2839	11.3792	81/16	–
$n = 4$	-92.91	-56.90	36.56	$\frac{15625}{1728}$	$n = 4$	-106.896	-46.2379	47.4048	729/64

[Retrieved from arXiv: 2404.08591 [hep-ph] Table. 3]

- coefficients $T_{n,m}$ for $N_f = 4$ (left) and $N_f = 3$ (right).
- The four-loop results are given in [6].
- the five-loop coefficient $T_{4,0}$ is from [7].
- $T_{4,1}$, $T_{4,2}$, and $T_{4,3}$ are generated from the RG analysis of [2] via the four-loop ($N_f = 4$ and $N_f = 3$) $\overline{\mathrm{MS}}$-scheme β function.

[6] M. R. Ahmady et al., Phys. Rev. D 67, 034017 (2003)

[7] P. A. Baikov, K. G. Chetyrkin, and J. H. K ühn, Phys. Rev. Lett. 101, 012002 (2008)

Backup slides: QCD parameters used

Parameter	Value	Source
α	$1/137.036$	PDG2022
$\alpha_s\,(M_\tau)$	0.312 ± 0.015	PDG2022
$m_u(2\,\mathrm{GeV})$	$2.16^{+0.49}_{-0.26}\,\mathrm{MeV}$	PDG2022
$m_d(2\,\mathrm{GeV})$	$4.67^{+0.48}_{-0.17}\,\mathrm{MeV}$	PDG2022
$m_s(2\,\mathrm{GeV})$	$\left(0.0934^{+0.0086}_{-0.0034}\right)\ \mathrm{GeV}$	PDG2022
f_π	$(0.13056 \pm 0.00019)\,/\sqrt{2}$ GeV	PDG2022
$m_n\langle\bar{n}n\rangle$	$-\frac{1}{2}f_\pi^2 m_\pi^2$	Phys. Rev. 175, 2195 (1968)
$m_s\langle\bar{s}s\rangle$	$r_m r_c m_n\langle\bar{n}n\rangle$	Phys. Rev. D 103, 114005 (2021)
$r_c \equiv \langle\bar{s}s\rangle/\langle\bar{n}n\rangle$	0.66 ± 0.10	Phys. Rev. D 103, 114005 (2021)
$m_s/m_n = r_m$	$27.33^{+0.67}_{-0.77}$	PDG2022
$\langle\alpha G^2\rangle$	$(0.0649 \pm 0.0035)\ \mathrm{GeV}^4$	Nucl. Phys. A 1039, 122743 (2023)
κ	3.22 ± 0.5	Nucl. Phys. A 1039, 122743 (2023)
$\alpha_s\langle\bar{n}n\rangle^2$	$\kappa\left(1.8 \times 10^{-4}\right)\ \mathrm{GeV}^6$	Phys. Rev. D 103, 114005 (2021)
$\alpha_s\langle\bar{s}s\rangle^2$	$r_c^2\alpha_s\langle\bar{n}n\rangle^2$	Phys. Rev. D 103, 114005 (2021)

Here, $m_n = (m_u + m_d)\,/2$ and $\langle\bar{n}n\rangle = \langle\bar{u}u\rangle = \langle\bar{d}d\rangle$.

[Retrieved from 2404.08591 [hep-ph] Table. 2]

Backup slides: Running of α_s

Running of α_s through flavour threshold is not taken into consideration as the effect will be overwhelmed by other uncertainties.

[See *K.G. Chetyrkin, B.A. Kniehl, M. Steinhauser, hep-ph/9708255 v2* **&** *T. G. Steele and V. Elias, Mod. Phys. Lett. A 13, 3151 (1998)*]

Backup slides: Optimization and Result

Flavour	s_0^{opt} $\left(\mathrm{GeV}^2\right)$	a_μ^{QCD} (lower bound)	a_μ^{QCD} (upper bound)
u	1.09	$\geq 472.7 \times 10^{-10}$	$\leq 567.2 \times 10^{-10}$
d	1.09	$\geq 118.1 \times 10^{-10}$	$\leq 141.7 \times 10^{-10}$
s	1.19	$\geq 66.2 \times 10^{-10}$	$\leq 79.5 \times 10^{-10}$
Total	–	$\geq 657.0 \times 10^{-10}$	$\leq 788.4 \times 10^{-10}$

[Retrieved from *Phys.Rev.D 110 (2024) 1, 014046* Table. 3]

Backup slides: Charmonium and Bottomonium Supplement

Our QCD prediction for the light-quark contributions:

$$(657.0 \pm 34.8) \times 10^{-10} \leq a_{\mu}^{\mathrm{HVP,LO}} \leq (788.4 \pm 41.8) \times 10^{-10} \, .$$

Supplement our bounds with charmonium and bottomonium resonance contributions of

$$a_{\mu\,,\,\bar{c}c\,,\,\bar{b}b}^{\mathrm{HVP,LO}} = (7.93 \pm 0.19) \times 10^{-10}$$

from *[A. Keshavarzi, D. Nomura, and T. Teubner, Phys. Rev. D 101, 014029 (2020)]*

$$\Rightarrow (664.9 \pm 34.8) \times 10^{-10} \leq a_{\mu}^{\mathrm{HVP,LO}} \leq (796.3 \pm 41.8) \times 10^{-10}$$