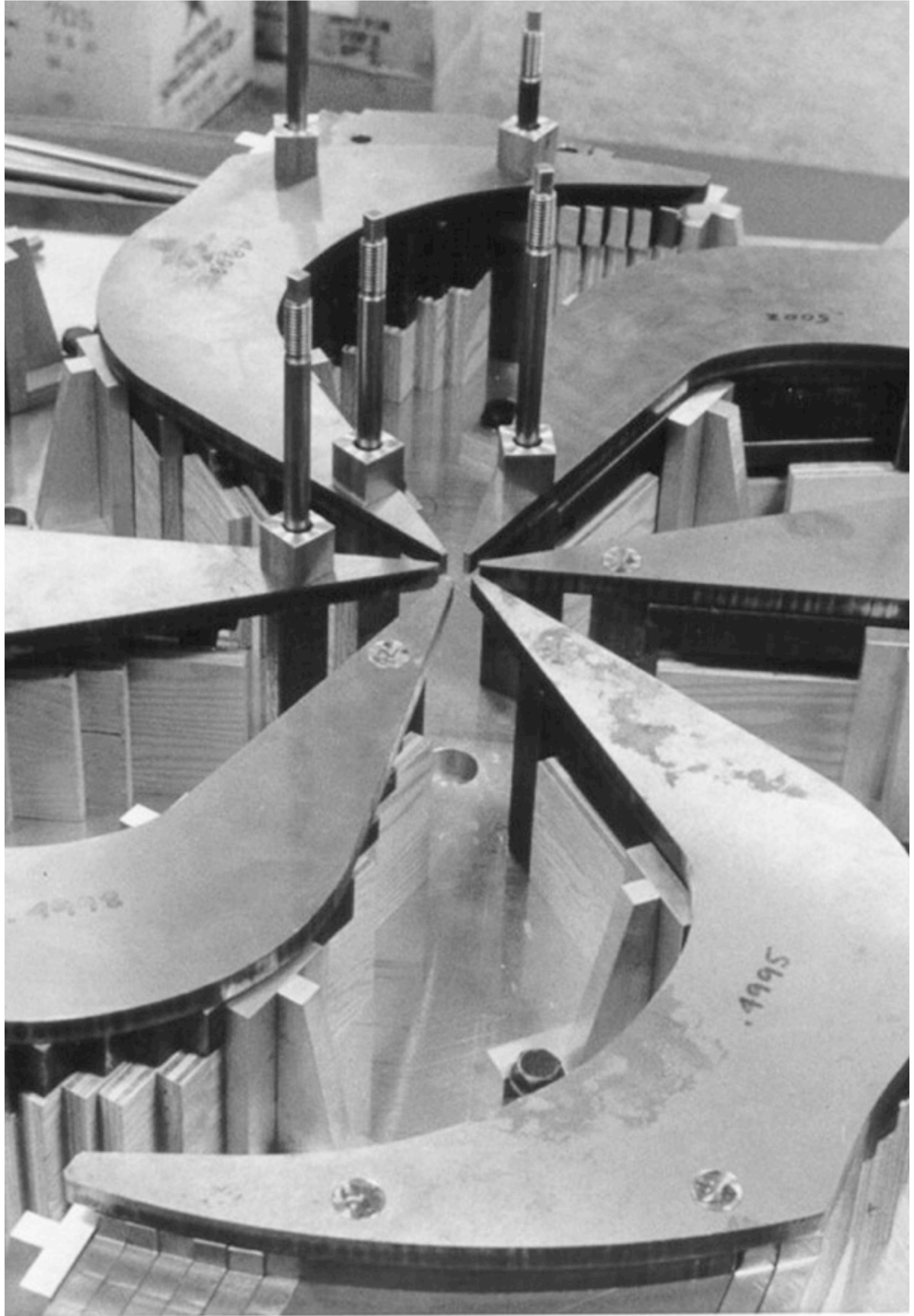
TRIUMF

Ab initio nuclear corrections to muonic atoms

Extracting nuclear radii from precision spectroscopy



Collaborators: Petr Navratil, Michael Gennari

Mehdi Drissi
TRIUMF - Theory department

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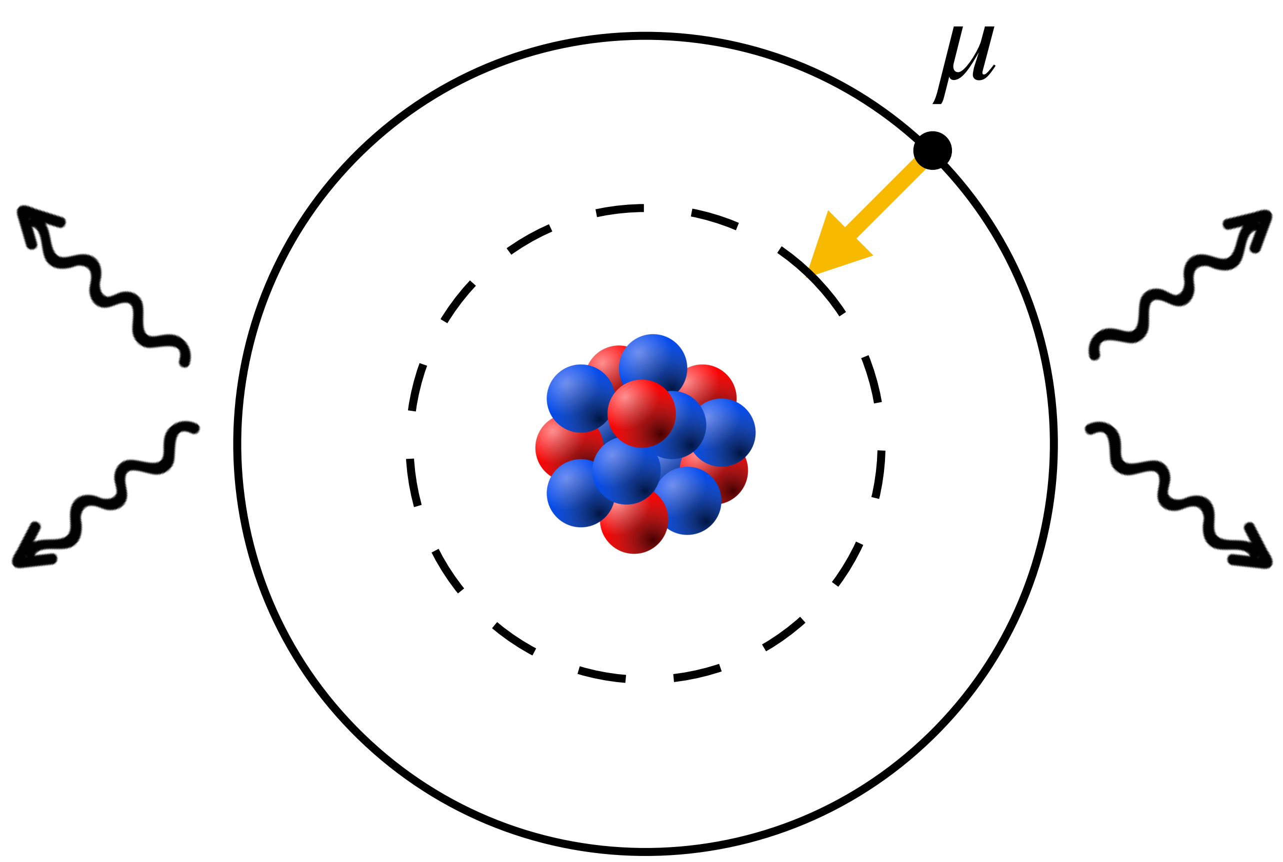
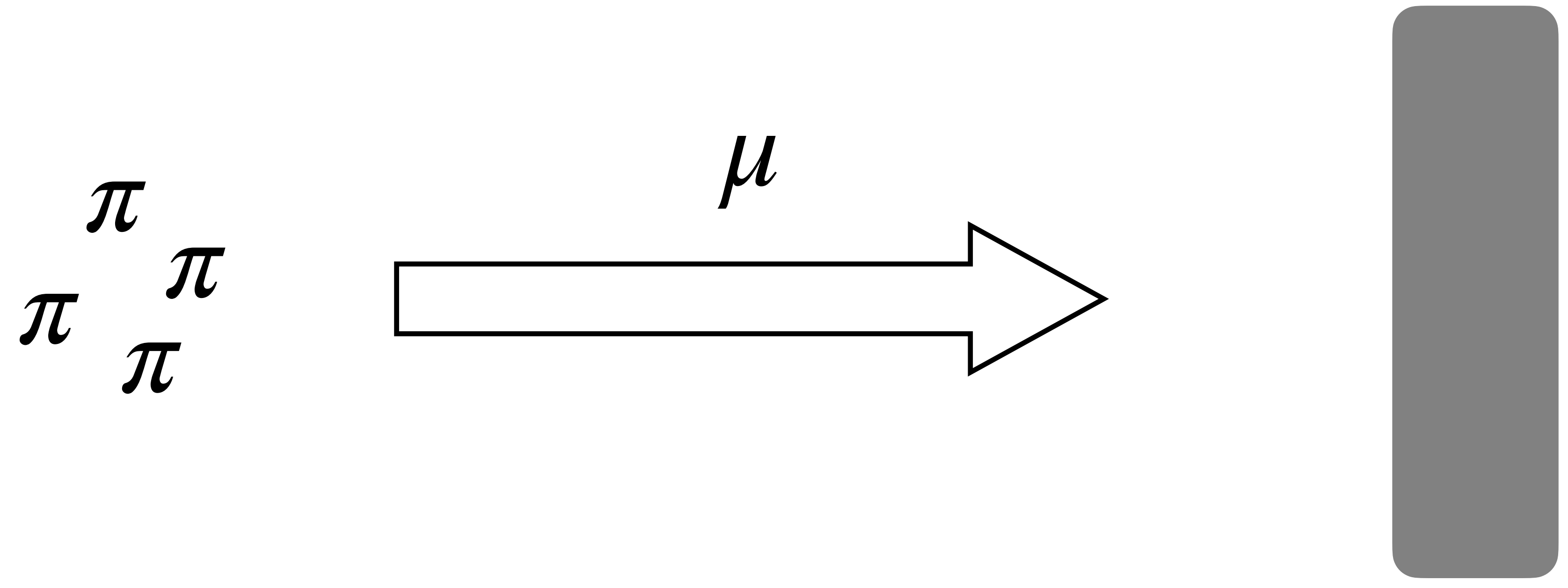
Discovery, accelerated

Muonic atoms and charge radii



Observing muonic atoms with X-rays

How to make muonic atom



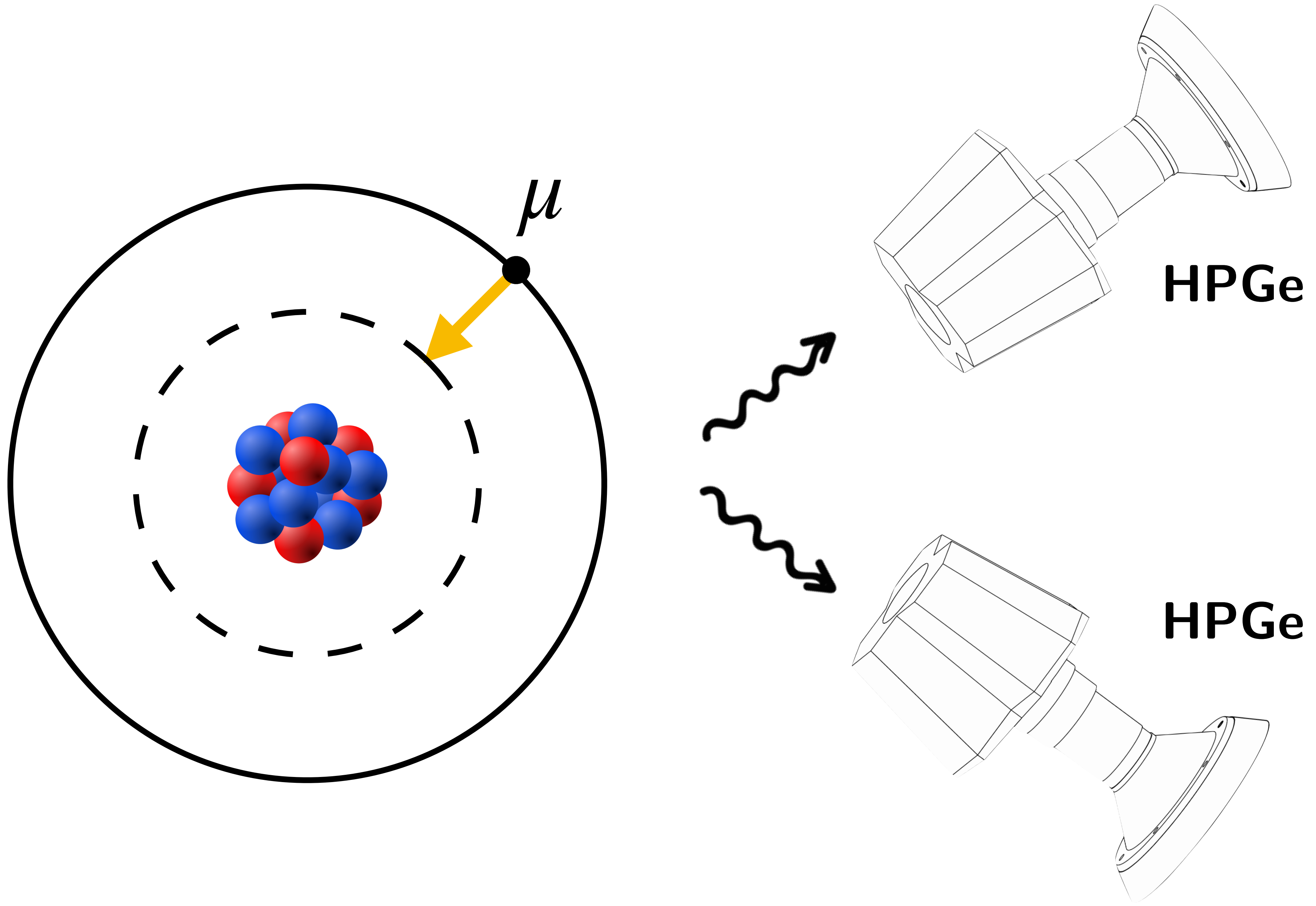
(i) Pion decay: muon source

(ii) High intensity beam: muon selection

(ii) Thick target: capture muons

Typically muons captured on orbitals with $n \sim \sqrt{\frac{m_\mu}{m_e}} \sim 14$

Observing characteristic X-rays



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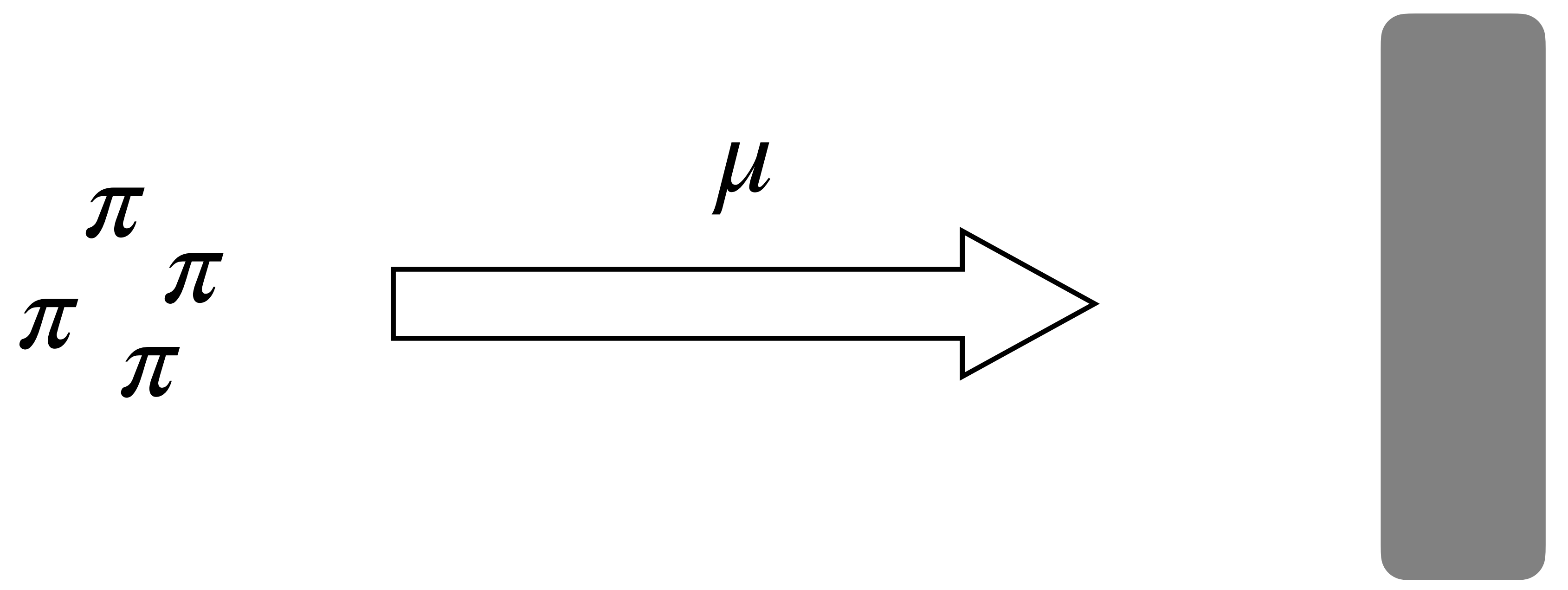

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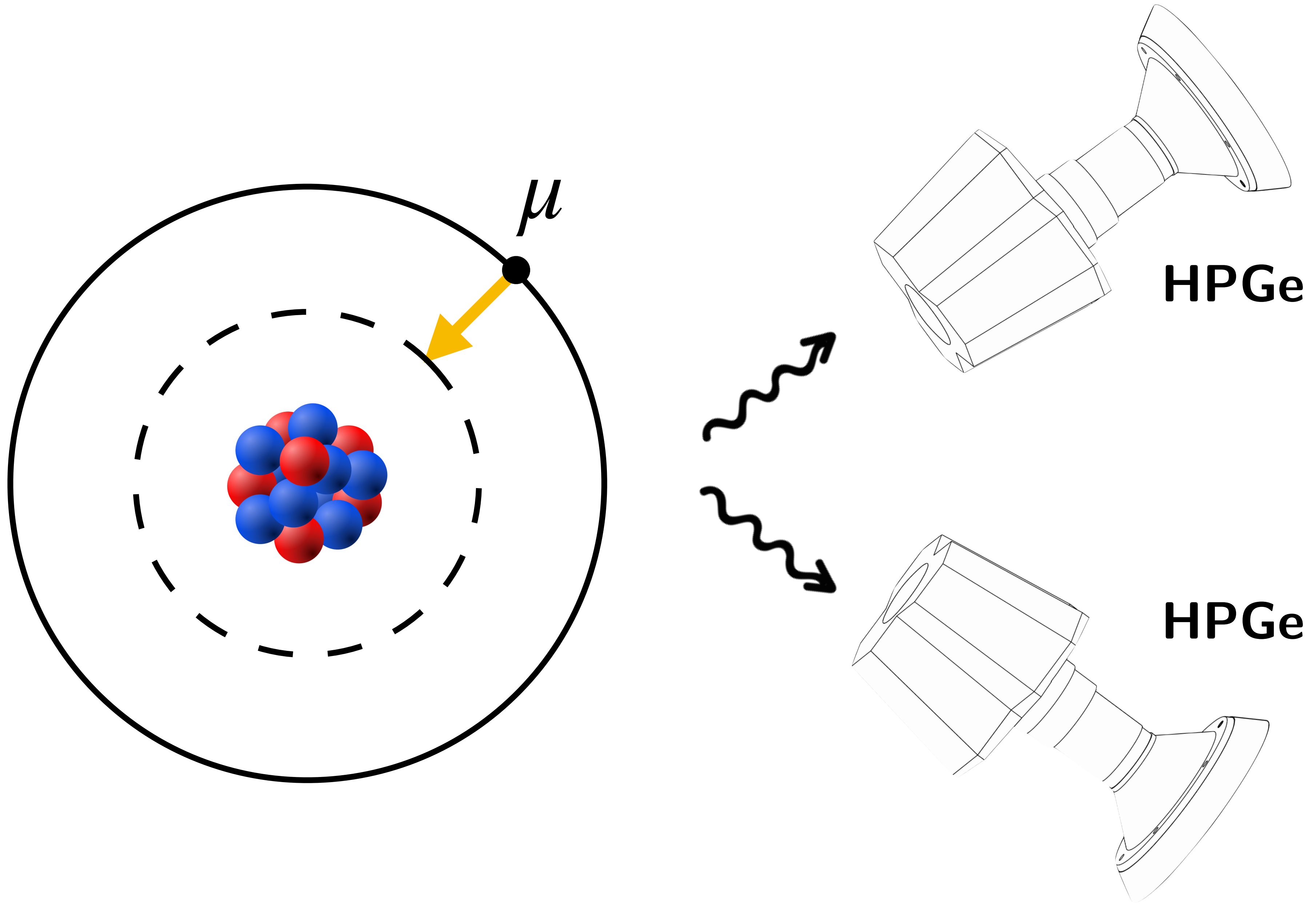

Muonic atom achievements

- Precise spectroscopy of almost all stable elements
- Charge radii extraction ⇒ **highest absolute accuracy**
- Combined with isotope-shift ⇒ **radii for unstable nuclei**

➡ **Higher sensitivity due to higher overlap** $\sim \left(\frac{m_\mu}{m_e}\right)^3 \sim 10^7$

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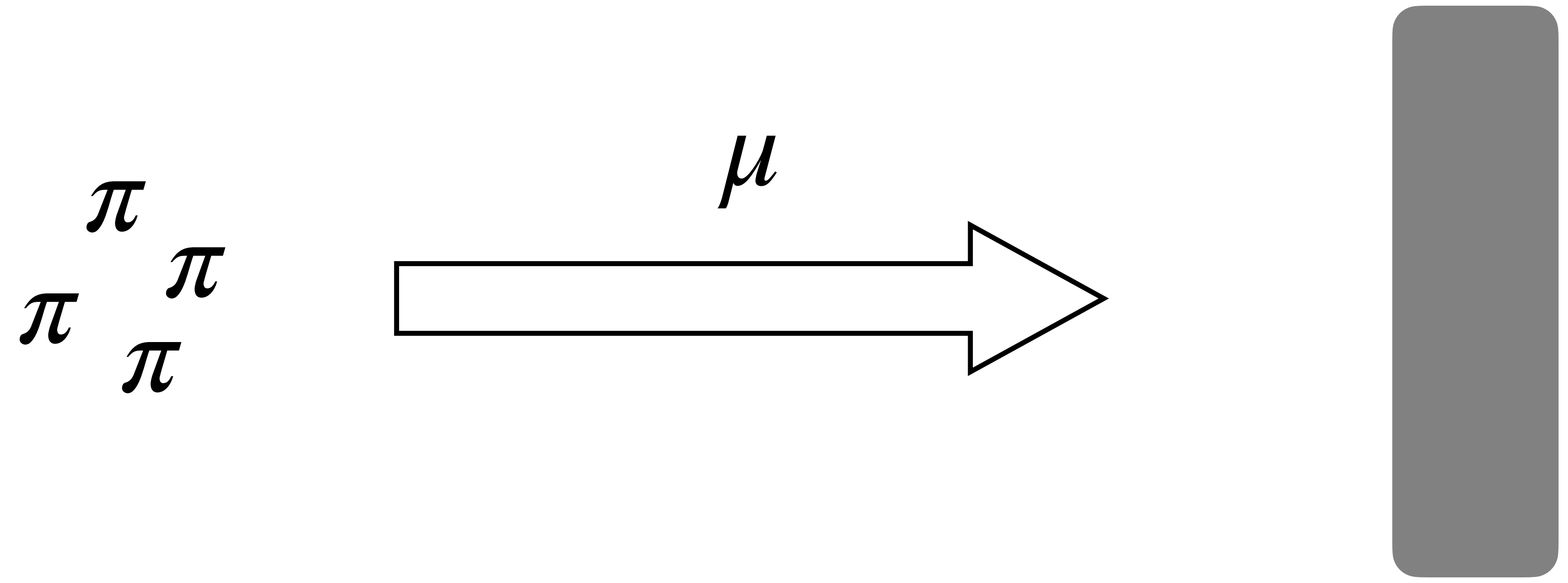

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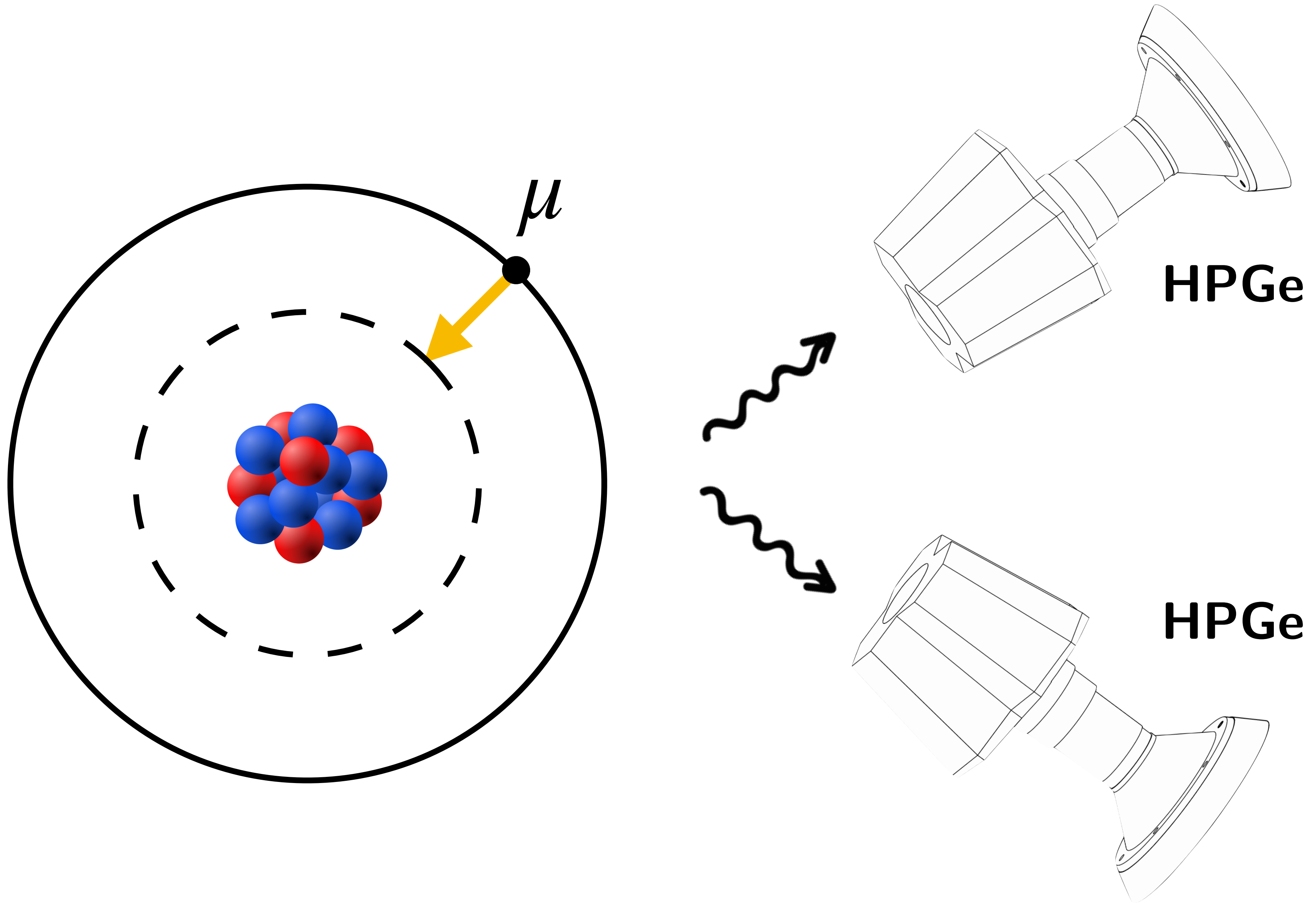

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Practical limitations

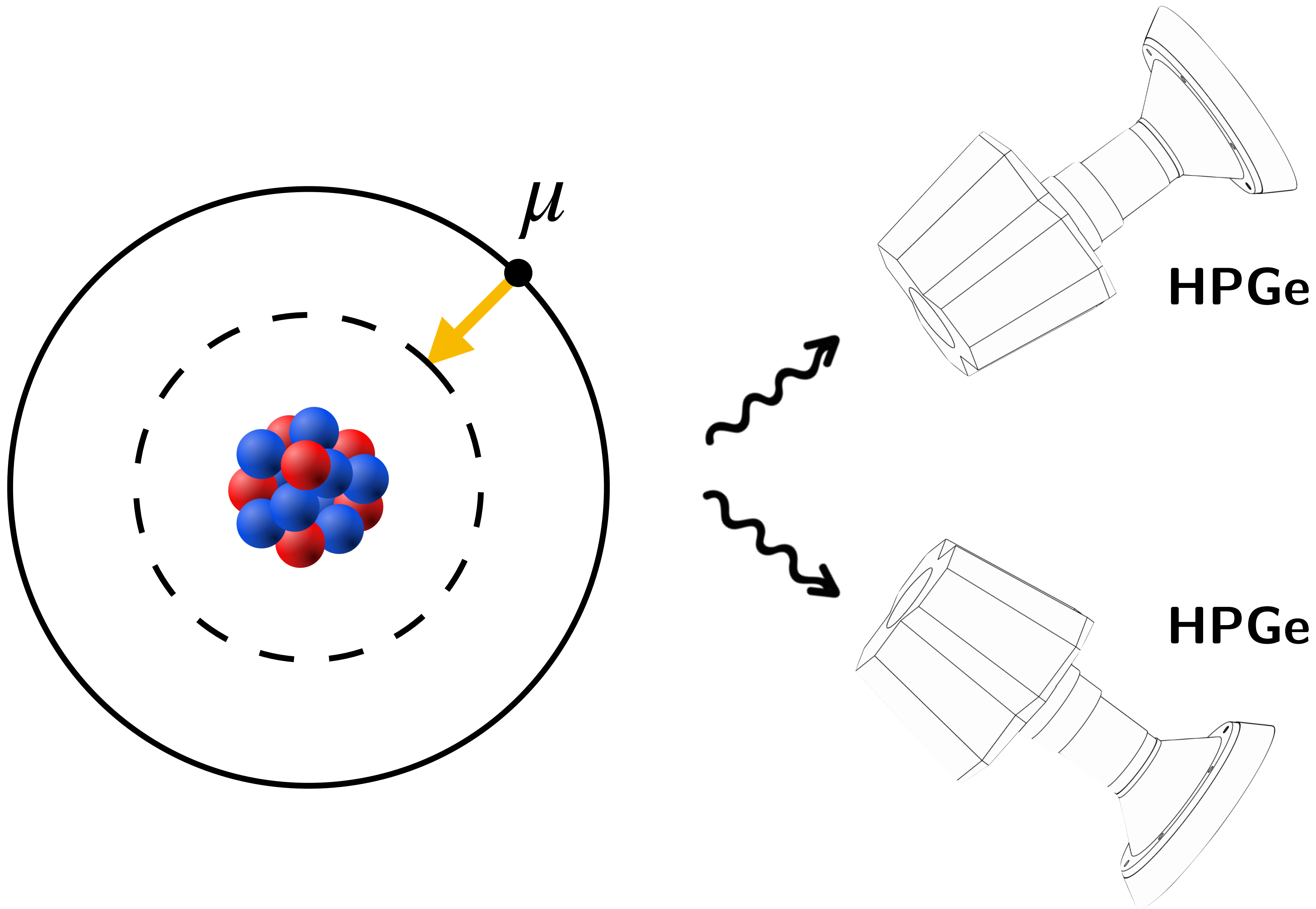
× In general: limitations are very experiment dependent

× Never with a perfect energy resolution

➡ **Many experimental challenges !**

Muonic atoms as a precision probe

Observing muonic atoms



QUARTET collaboration

- Improving energy resolution
  - Quantum sensor detector to reach low-Z nuclei
  - **On-going work at PSI with ^{6}Li / ^{7}Li target**

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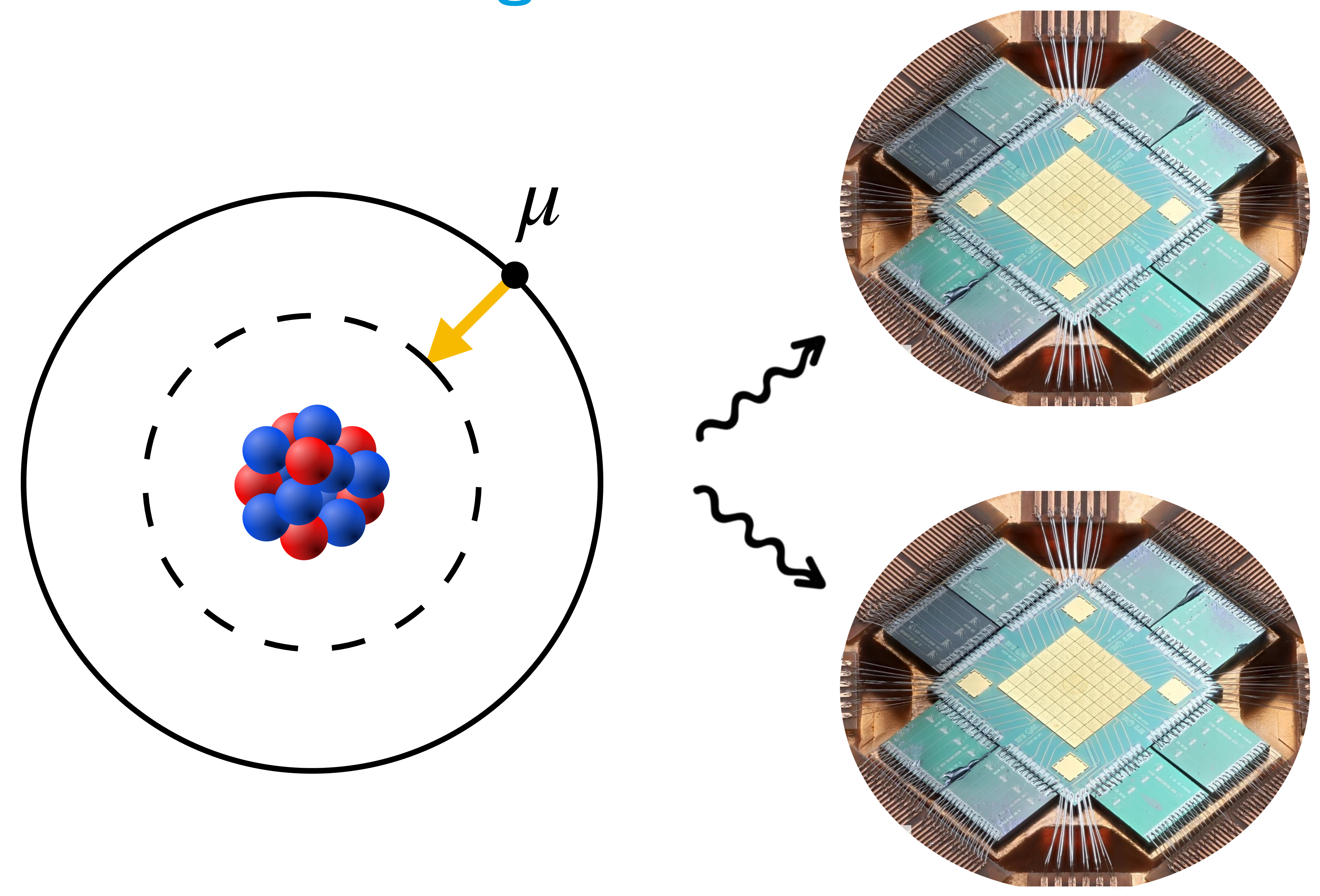

[Unger et al. J. Low Temp. Phys. (2024)]

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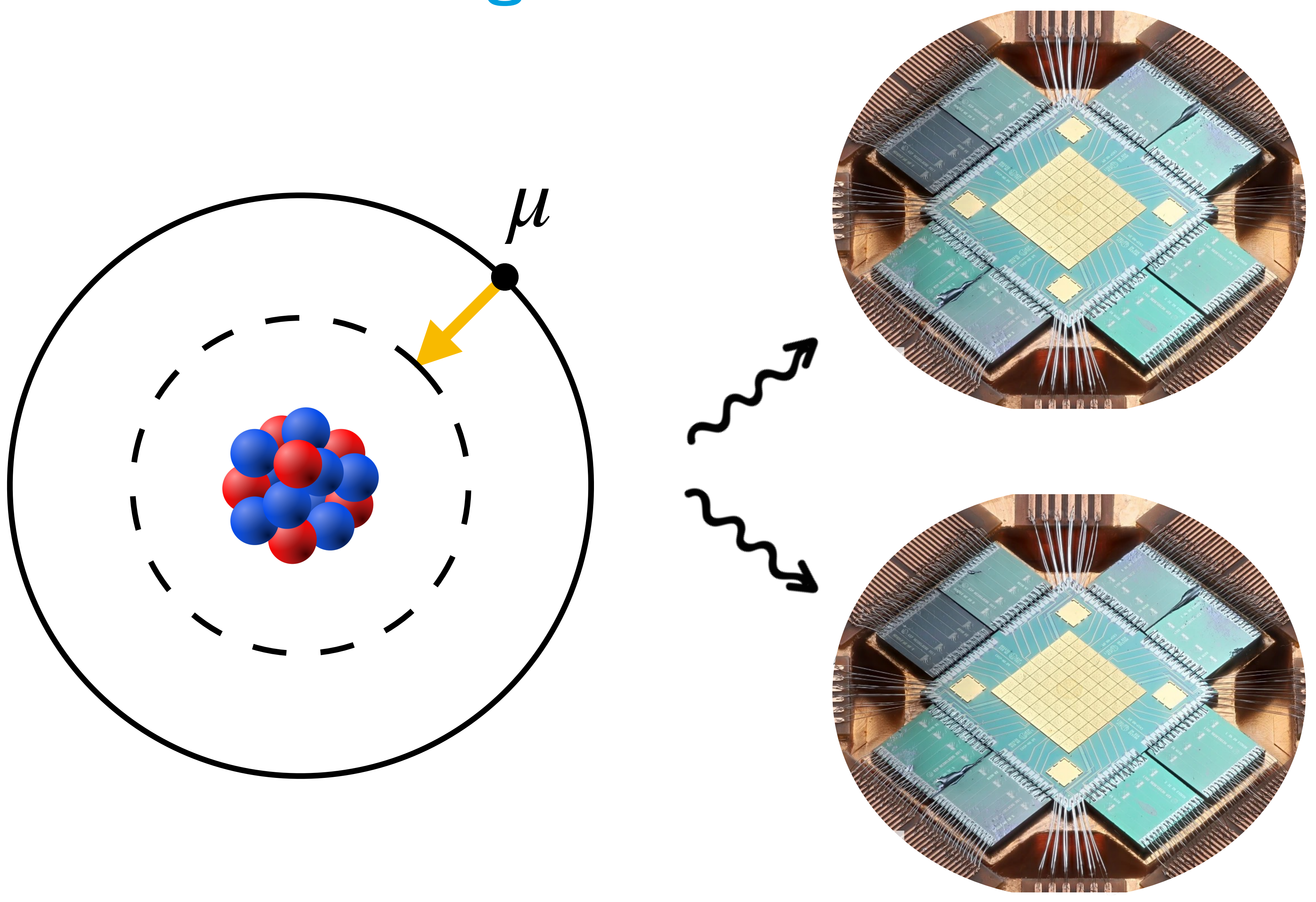

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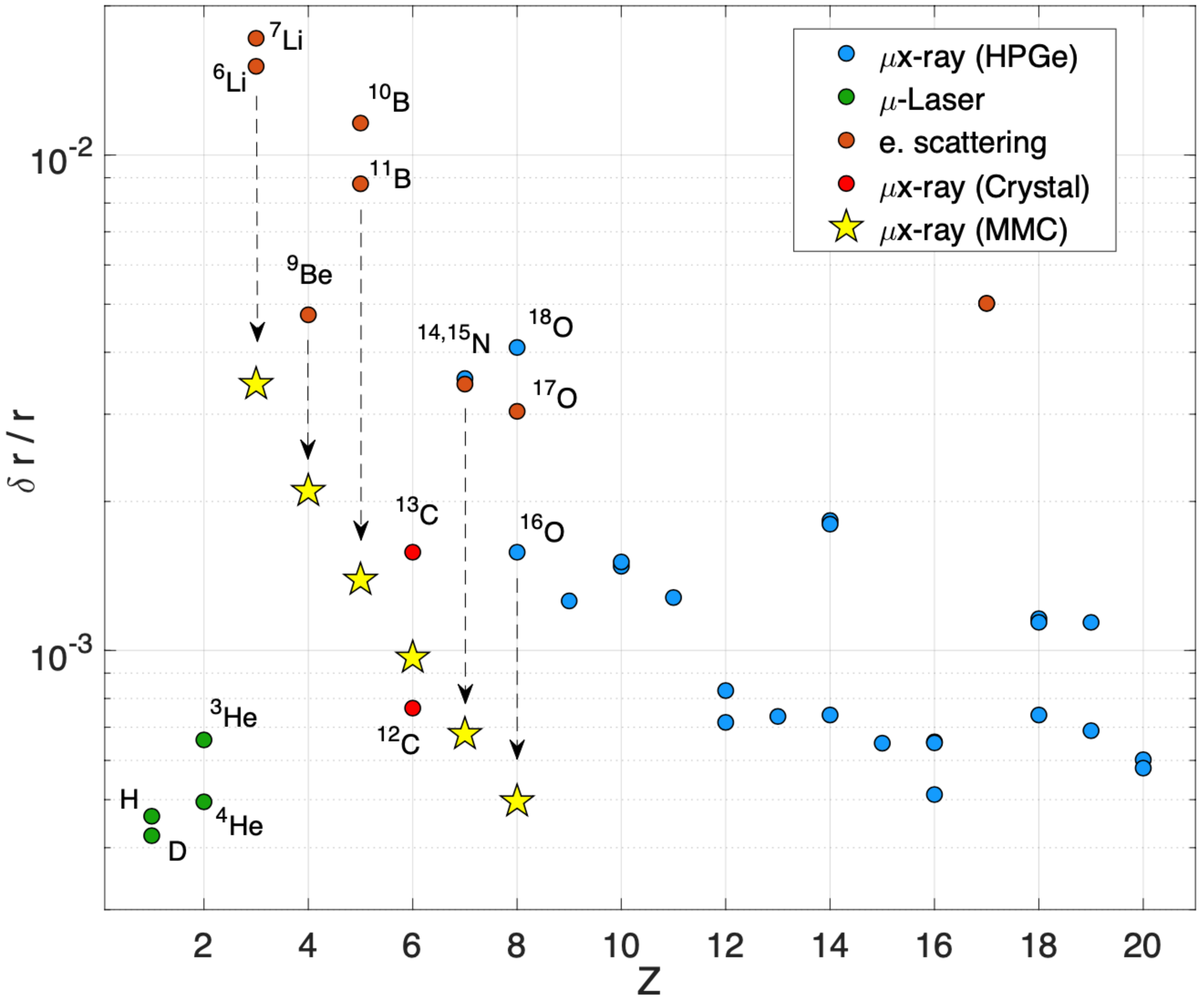

[Antognini et al, arXiv:2210.16929] NuPECC Long Range Plan 2024

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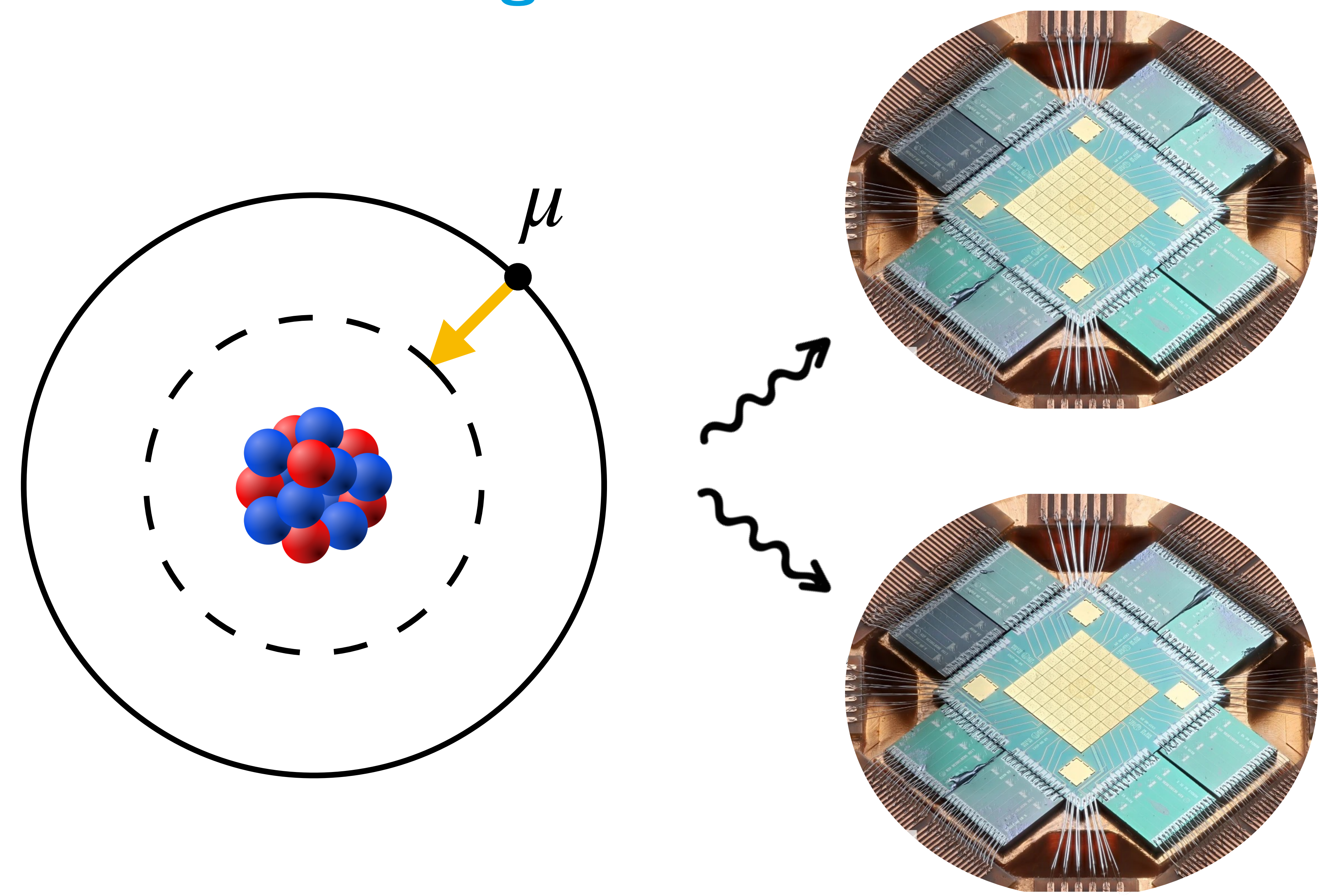

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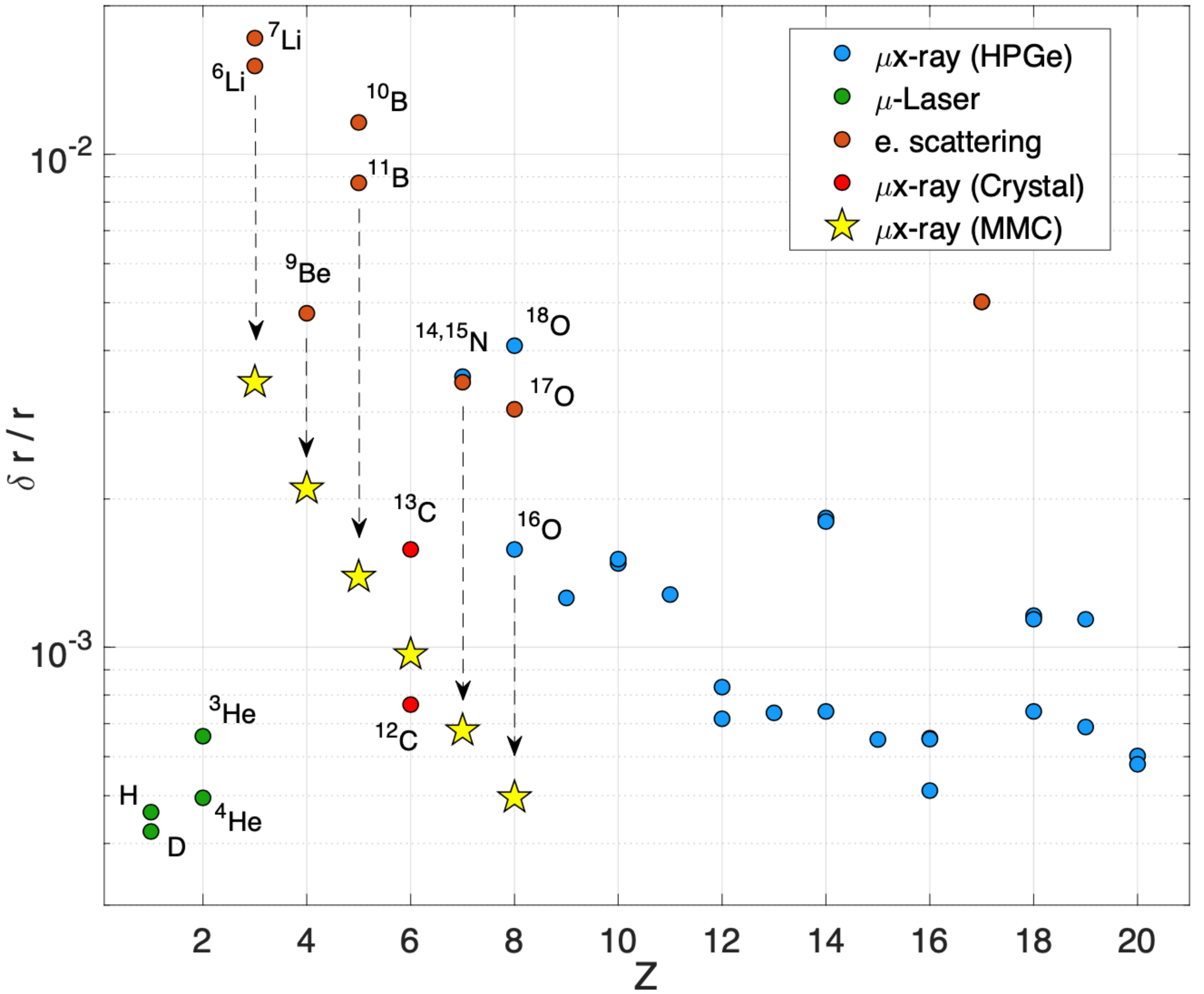

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Theoretical challenge: reach 10 meV uncertainty!

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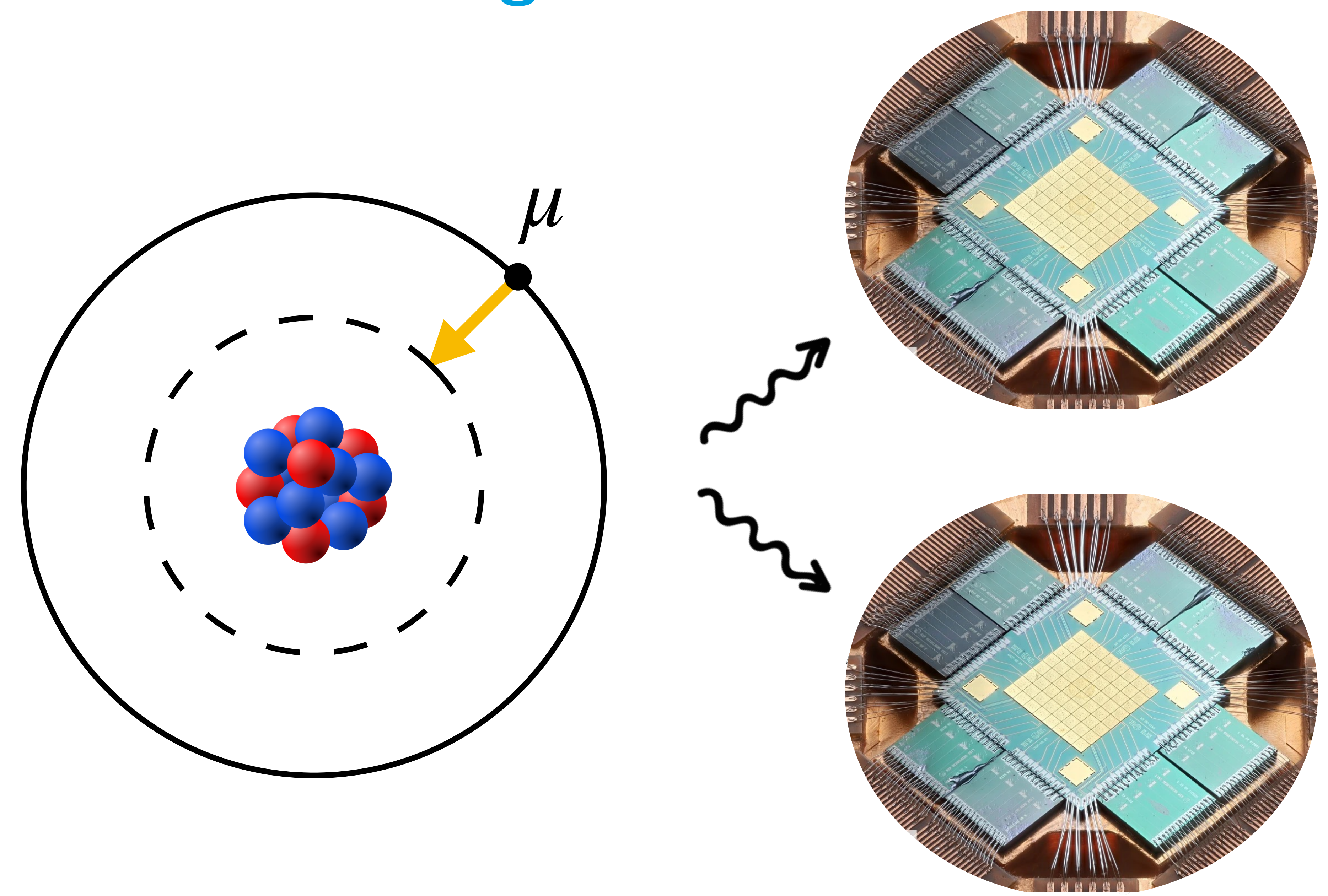

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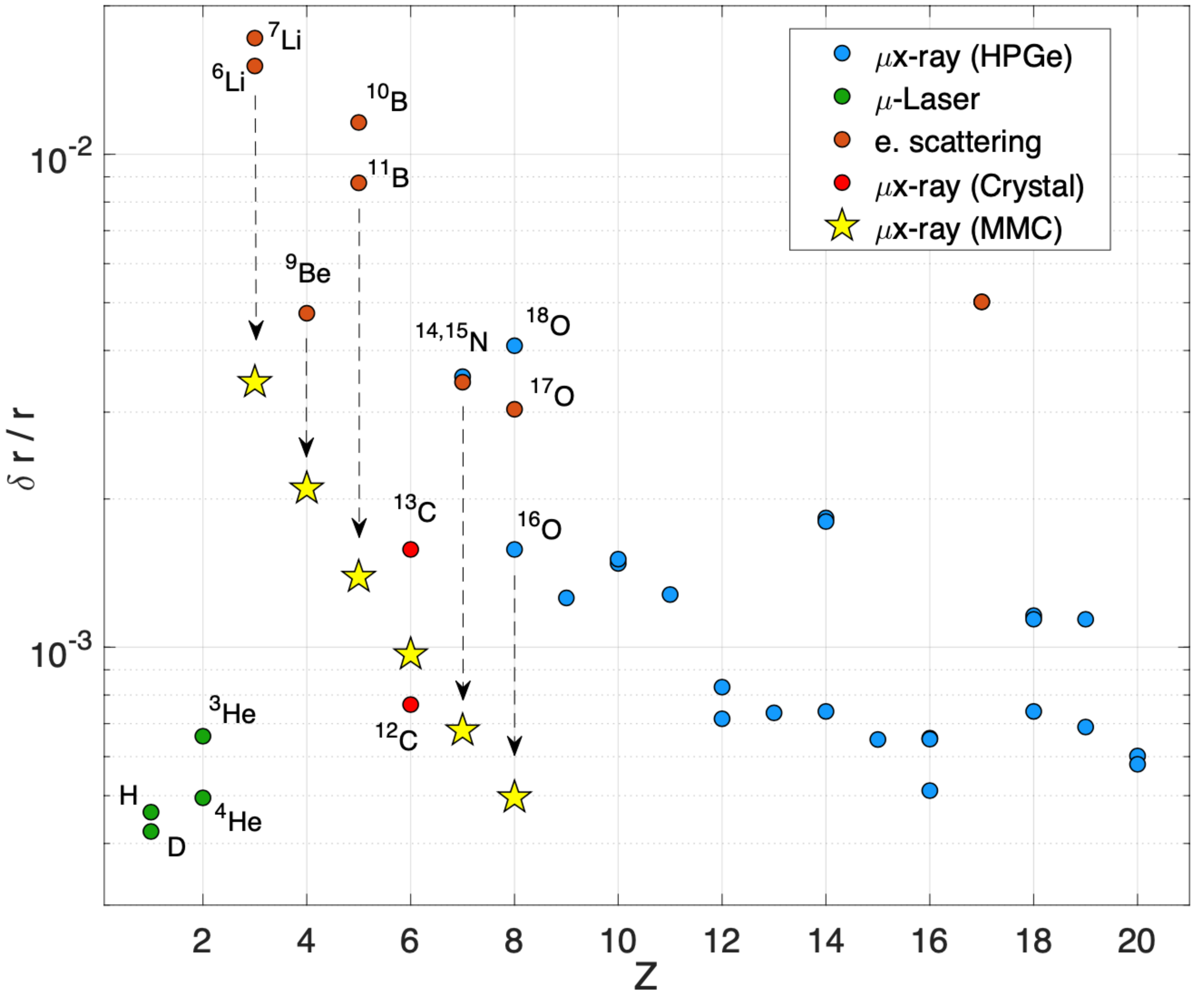

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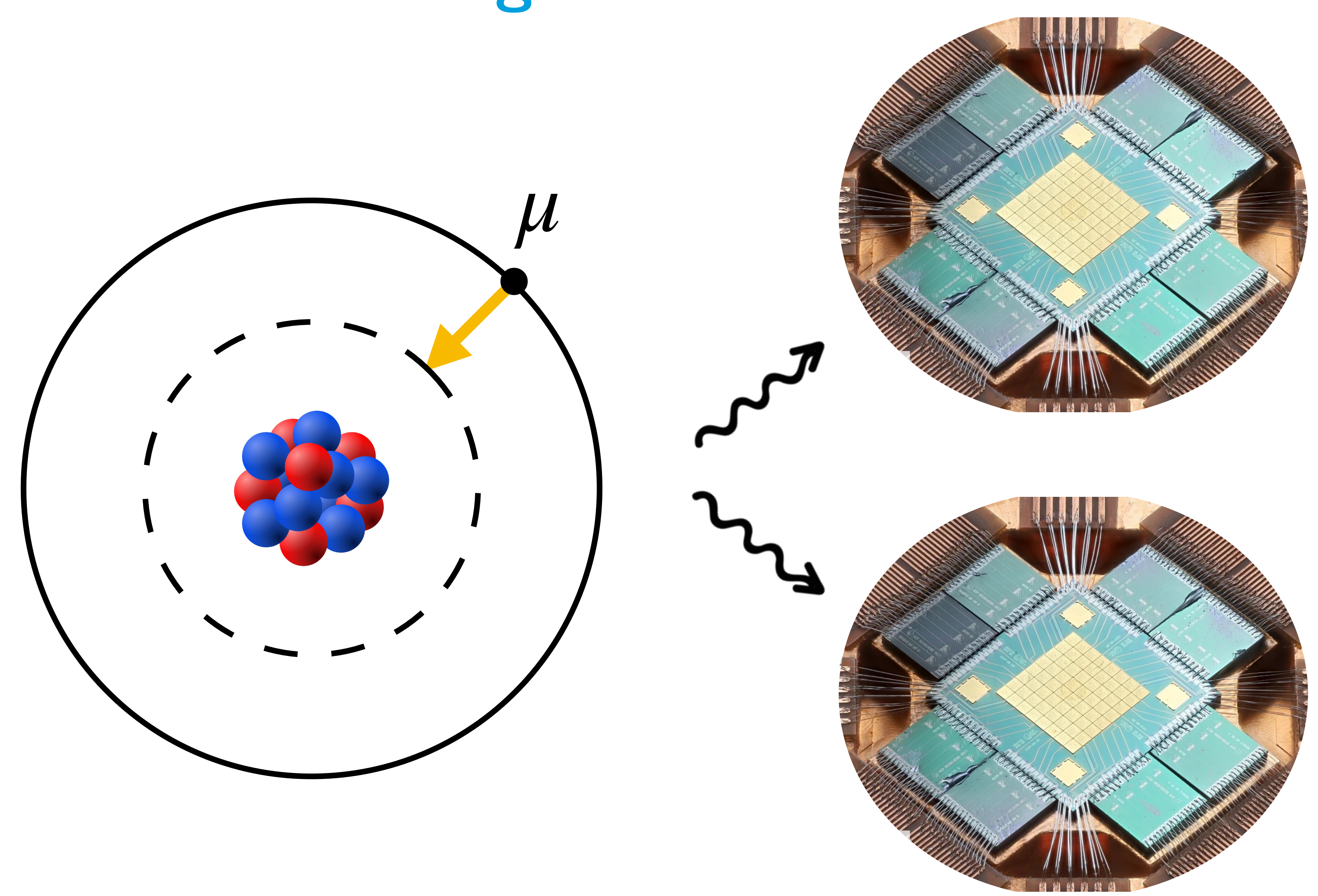

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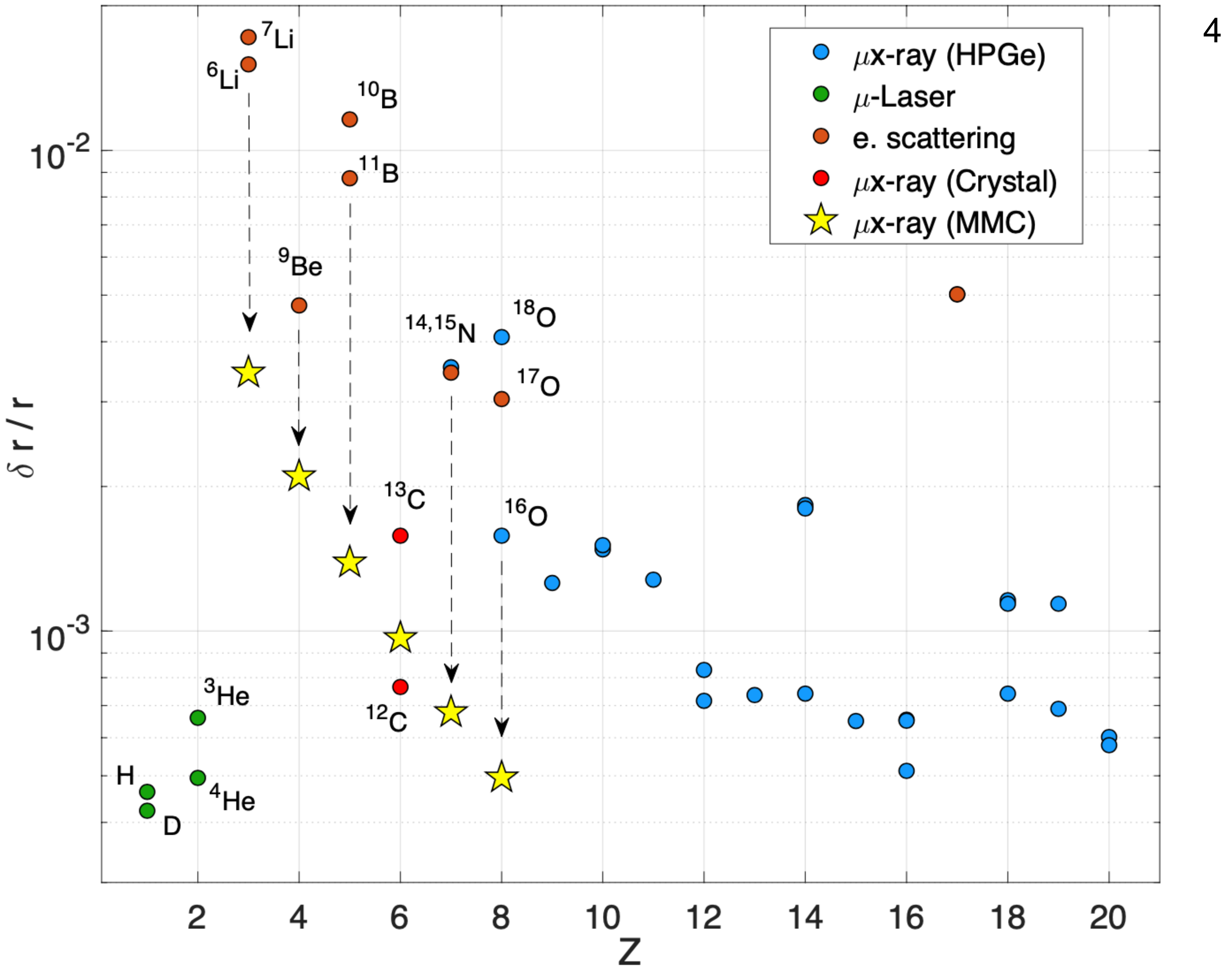

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On-going puzzles: $\sim 3.5\sigma$ for $^{3-4}$He isotope shift

From energy levels to nuclear structure

Converting experimental data

- What to do once precise value of energy levels is known ?
  - Can be used to **test fundamental constants** like R_∞, α, m_e
  - Can be used to extract **nuclear structure information** like r_c
  - Can be used to test validity of **many-body calculations**

- Example in practice: Lamb shift in meV $2S_{1/2} - 2P_{1/2}$ (r_x in fm)

  [Antognini et al, SciPost (2021)]

$$\Delta E(\mu\mathrm{H}) = 206.0336(15) - 5.2275(10) \times r_p^2 + 0.0332(20)$$
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- Main degrees of freedom
  - Nucleons $\rightarrow N$; photon $\rightarrow A$; Muon $\rightarrow \psi_\mu$

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  - For simplicity assume non-relativistic nucleons of equal mass

$$H = H_{Nucl} + e \int \mathrm{d}^3x \; J_\mu(x) A^\mu(x)$$
$$+ \frac{e^2}{2m} \int \mathrm{d}^3x \mathrm{d}^3y \; f_{SG}(x,y) \; \vec{A}(x).\vec{A}(y)$$
$$+ H_{QED}$$

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- General approach to compute bound state of H
  - ✗ In principle use Bethe-Salpeter $\Rightarrow$ bound states $\equiv G_n$ poles
  - ✓ In practice use **effective external potential**
  - Corrections up to $(Z\alpha)^5$ to match exp accuracy

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Radius extraction master formula

$$\delta_\mathrm{LS} = \delta_\mathrm{QED} + \mathscr{C}\ r_c^2 + \delta_\mathrm{NS}$$

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Simple point-like nucleus → δ_QED

Finite nucleus size effect → $\mathscr{C}\ r_c^2$

Nuclear structure dependent → δ_NS

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Bound states QED contributions

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Bound muon within potential

- Zero-order: external Coulomb potential
  - Solve exactly for $H_0 = \frac{\vec{p}^2}{2m_r} - \frac{Z\alpha}{r}$
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- What effective potential to apply on muon ?
  - **Effective potential** as perturbation away from Coulomb
  - Defined to **match QED** scattering at a given order
  - Bound-state ⇒ **Distorted Wave Born approximation**

$$E_{nl} = E^{(0)} + \langle V^{(1)} \rangle + \langle V^{(2)} \rangle + \langle V^{(1)} \frac{1}{(E_0 - H_0)'} V^{(1)} \rangle + \ldots$$

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  - Electron vacuum polarization: $a_\mu \sim \lambda_e$ ⇒ **main one!**
  - Finite nuclear mass ⇒ recoil and relativistic corrections
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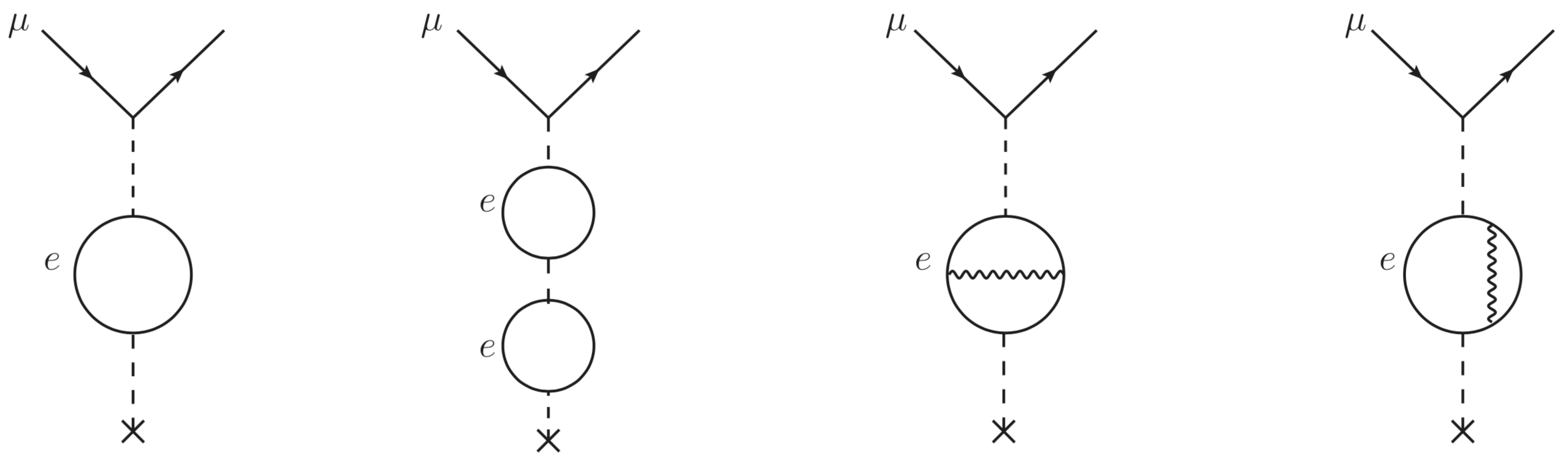
Example: electron vacuum polarization corrections

[Pachucki et al. Review of Modern Physics (2024)]



$\Rightarrow \delta_{\text{QED}}$ term in δ_{LS}

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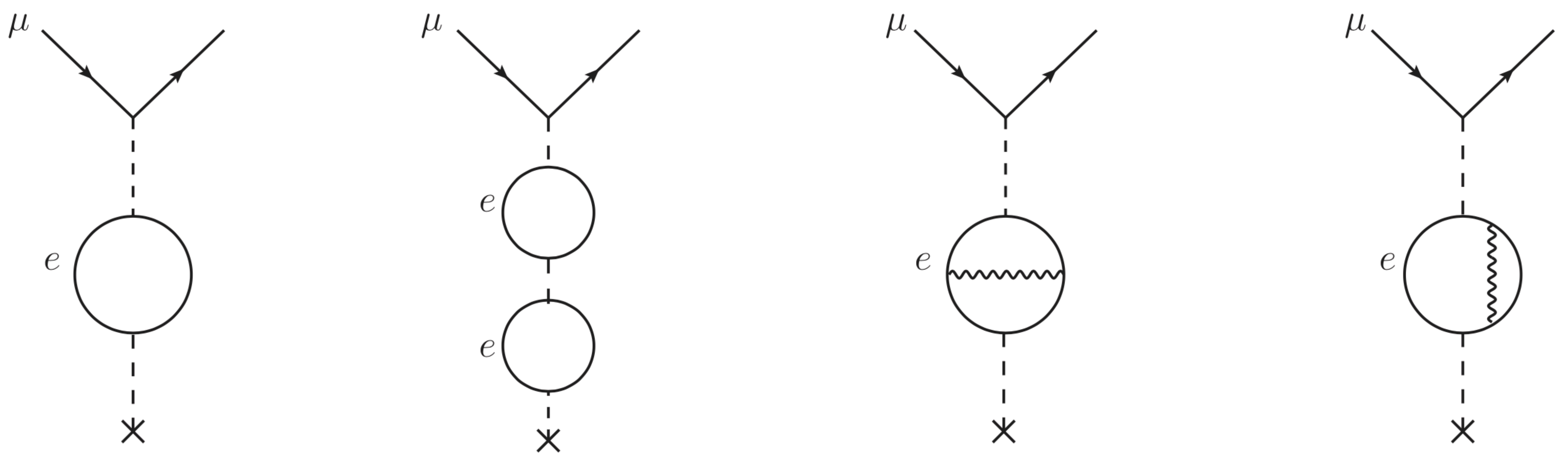

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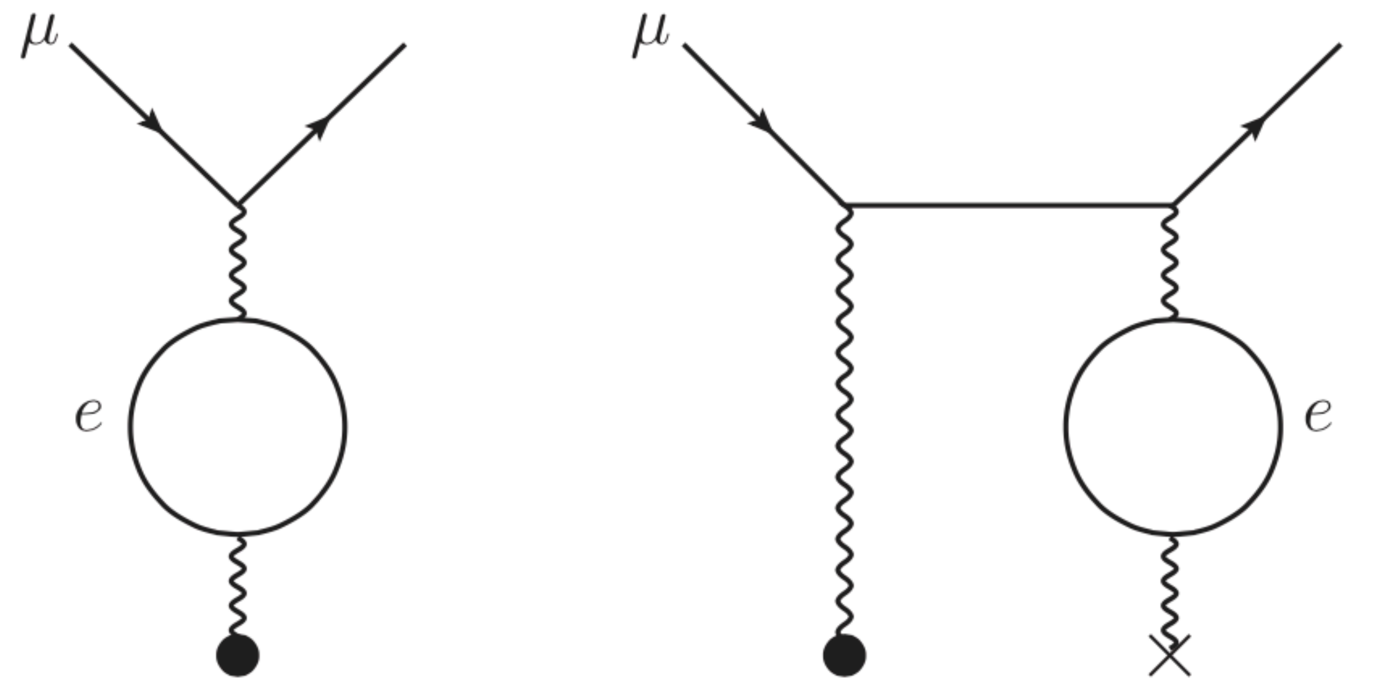

$\Rightarrow \mathscr{C} r_c^2$ term in δ_{LS}

Bound states QED contributions

Section	Order	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
III.A	$\alpha(Z\alpha)^2$	eVP$^{(1)}$	205.007 38	227.634 70	1641.886 2	1665.773 1
III.A	$\alpha^2(Z\alpha)^2$	eVP$^{(2)}$	1.658 85	1.838 04	13.084 3	13.276 9
III.A	$\alpha^3(Z\alpha)^2$	eVP$^{(3)}$	0.007 52	0.008 42(7)	0.073 0(30)	0.074 0(30)
III.B	$(Z, Z^2, Z^3)\alpha^5$	Light-by-light eVP	−0.000 89(2)	−0.000 96(2)	−0.013 4(6)	−0.013 6(6)
III.C	$(Z\alpha)^4$	Recoil	0.057 47	0.067 22	0.126 5	0.295 2
III.D	$\alpha(Z\alpha)^4$	Relativistic with eVP$^{(1)}$	0.018 76	0.021 78	0.509 3	0.521 1
III.E	$\alpha^2(Z\alpha)^4$	Relativistic with eVP$^{(2)}$	0.000 17	0.000 20	0.005 6	0.005 7
III.F	$\alpha(Z\alpha)^4$	μSE$^{(1)}$ + μVP$^{(1)}$, LO	−0.663 45	−0.769 43	−10.652 5	−10.926 0
III.G	$\alpha(Z\alpha)^5$	μSE$^{(1)}$ + μVP$^{(1)}$, NLO	−0.004 43	−0.005 18	−0.174 9	−0.179 7
III.H	$\alpha^2(Z\alpha)^4$	μVP$^{(1)}$ with eVP$^{(1)}$	0.000 13	0.000 15	0.003 8	0.003 9
III.I	$\alpha^2(Z\alpha)^4$	μSE$^{(1)}$ with eVP$^{(1)}$	−0.002 54	−0.003 06	−0.062 7	−0.064 6
III.J	$(Z\alpha)^5$	Recoil	−0.044 97	−0.026 60	−0.558 1	−0.433 0
III.K	$\alpha(Z\alpha)^5$	Recoil with eVP$^{(1)}$	0.000 14(14)	0.000 09(9)	0.004 9(49)	0.003 9(39)
III.L	$Z^2\alpha(Z\alpha)^4$	nSE$^{(1)}$	−0.009 92	−0.003 10	−0.084 0	−0.050 5
III.M	$\alpha^2(Z\alpha)^4$	$\mu F_1^{(2)}$, $\mu F_2^{(2)}$, μVP$^{(2)}$	−0.001 58	−0.001 84	−0.031 1	−0.031 9
III.N	$(Z\alpha)^6$	Pure recoil	0.000 09	0.000 04	0.001 9	0.001 4
III.O	$\alpha(Z\alpha)^5$	Radiative recoil	0.000 22	0.000 13	0.002 9	0.002 3
III.P	$\alpha(Z\alpha)^4$	hVP	0.011 36(27)	0.013 28(32)	0.224 1(53)	0.230 3(54)
III.Q	$\alpha^2(Z\alpha)^4$	hVP with eVP$^{(1)}$	0.000 09	0.000 10	0.002 6(1)	0.002 7(1)

[Pachucki et al. Review of Modern Physics (2024)]

The two-photon exchange nuclear correction

Radius extraction master formula

$$\delta_{\text{LS}} = \delta_{\text{QED}} + \mathscr{C}\ r_c^2 + \delta_{\text{NS}}$$

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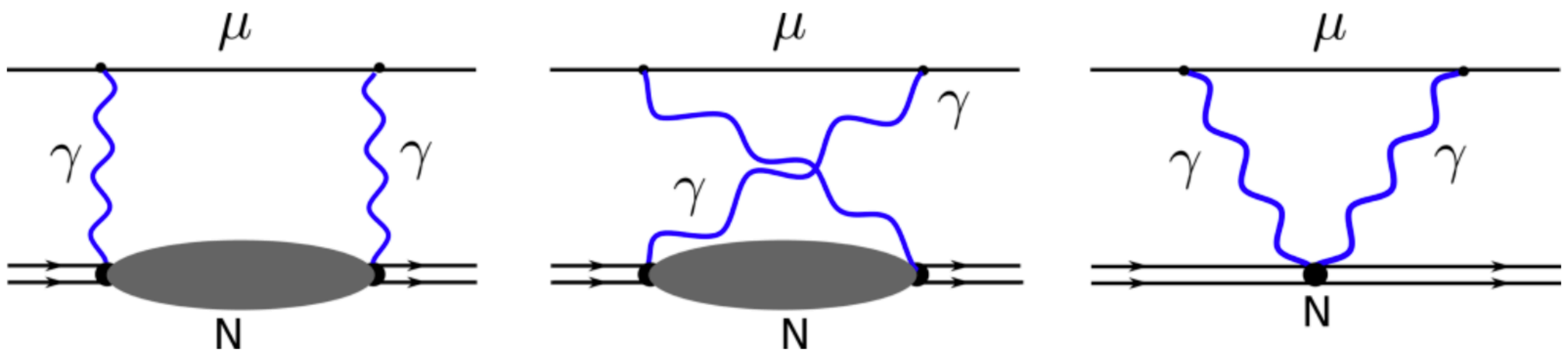

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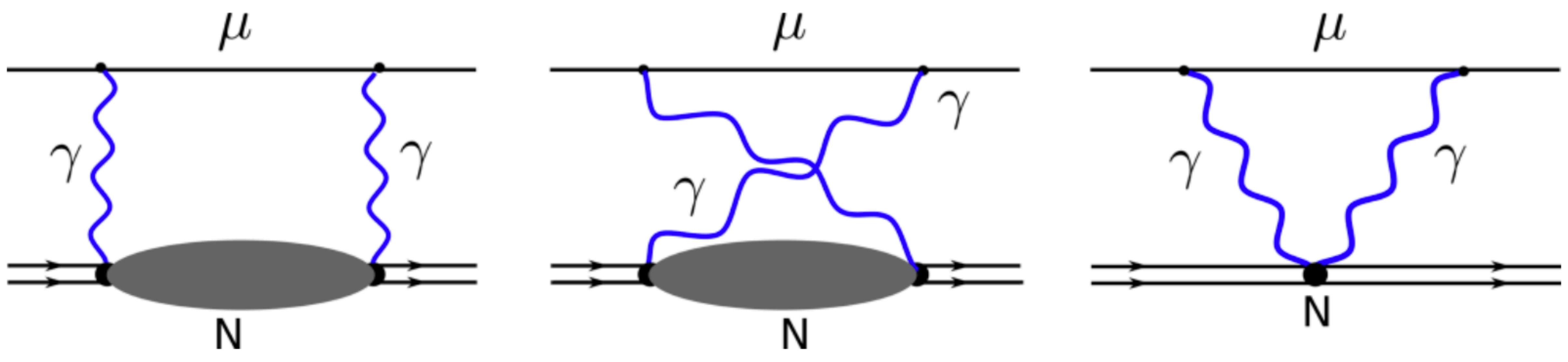

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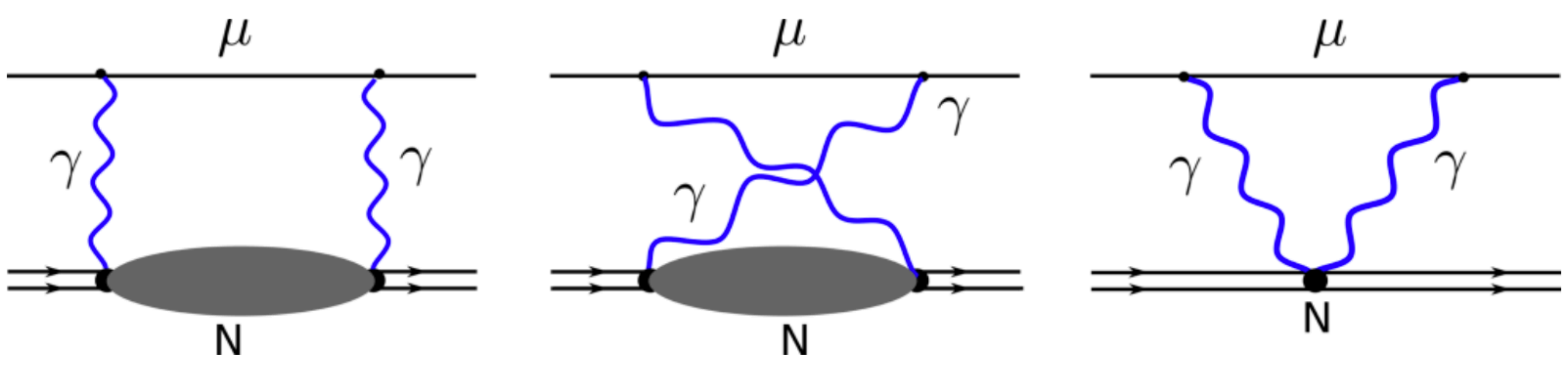

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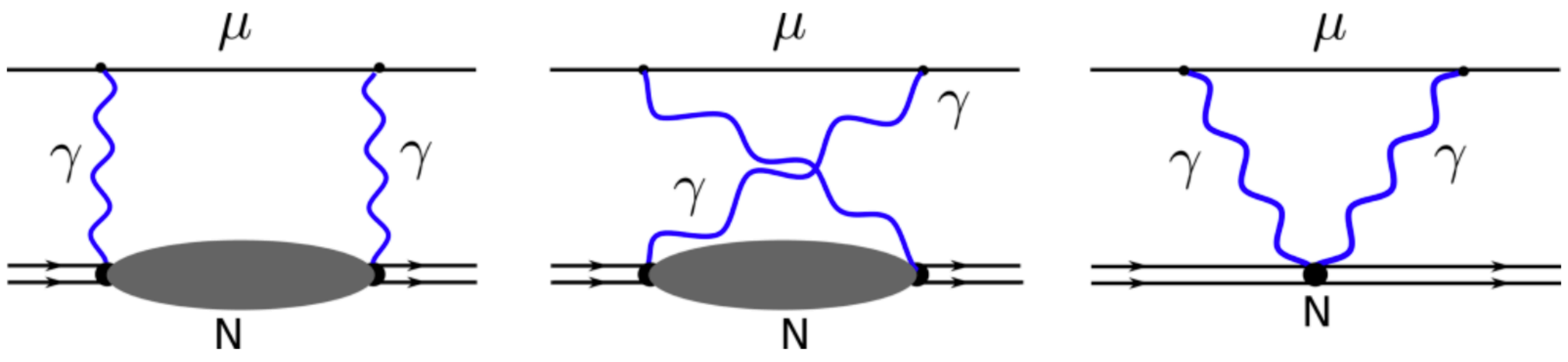

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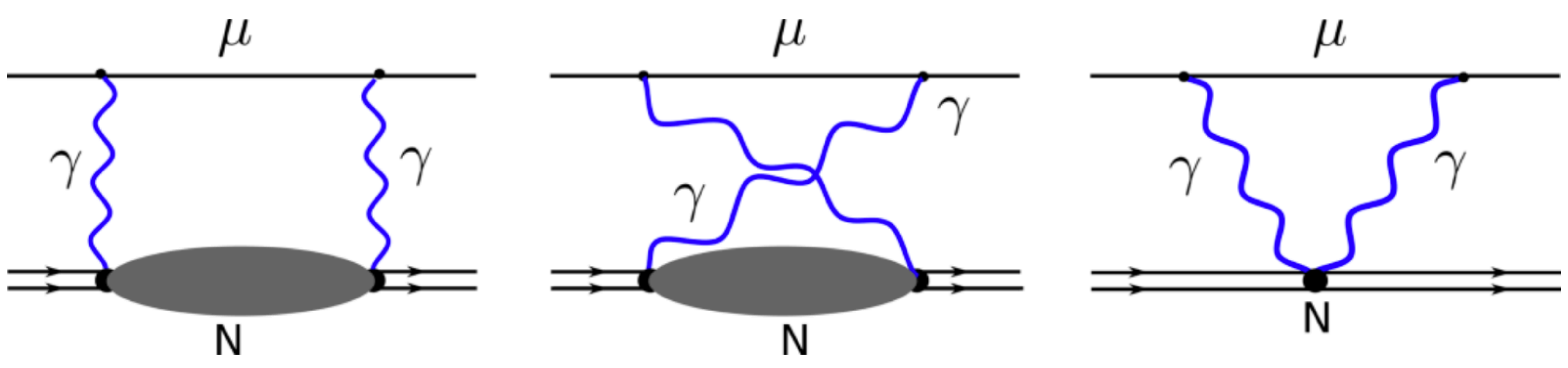

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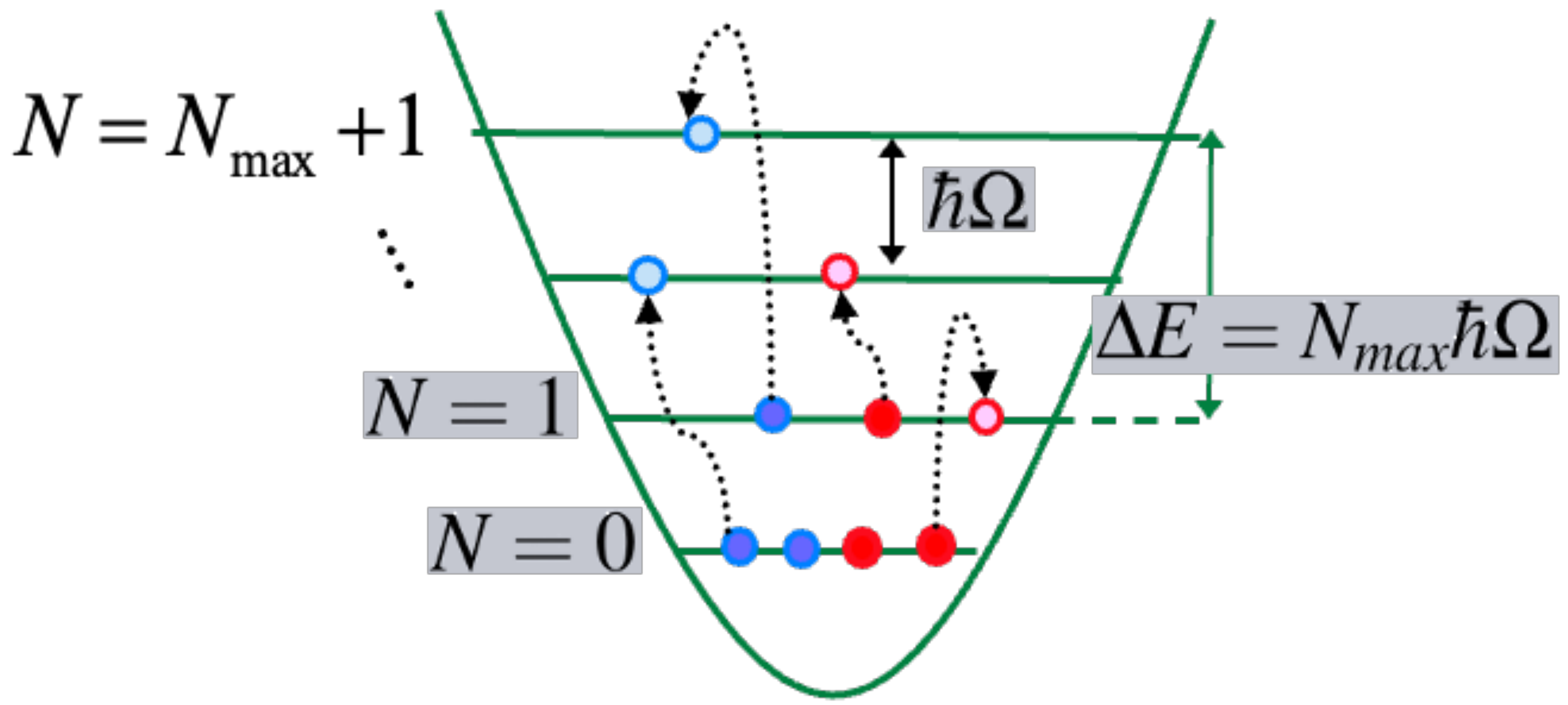
- Spectral functions:
  - ➡ $S_X(\omega,q) \equiv \sum_{J\geq 0} \sum_{N\neq 0} |\langle N | O_{X,J}(q) | \Psi \rangle|^2 \delta(E_N - E_0 - \omega)$

Ab initio nuclear corrections



Nuclear physics modelling

Model used for nuclear currents

- Electromagnetic current modelling
  - General one-body current for point-like particles
  - Form factors given by the isovector dipole model
  - $f_{SN}(q) = \left(1 + \frac{q^2}{M_V^2}\right)^{-2} \quad , \quad F_{1,2}^{(T)}(q) = F_{1,2}^{(T)}(0)\, f_{SN}(q)$

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- Ab initio nuclear interaction [Entem et al. (2017)] [Somà et al. (2020)]
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➡ **Estimate interaction uncertainty**

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- Model space
  - Harmonic oscillator Slater determinant
  - Vary many-body basis: $(\Omega, N_{\max})$

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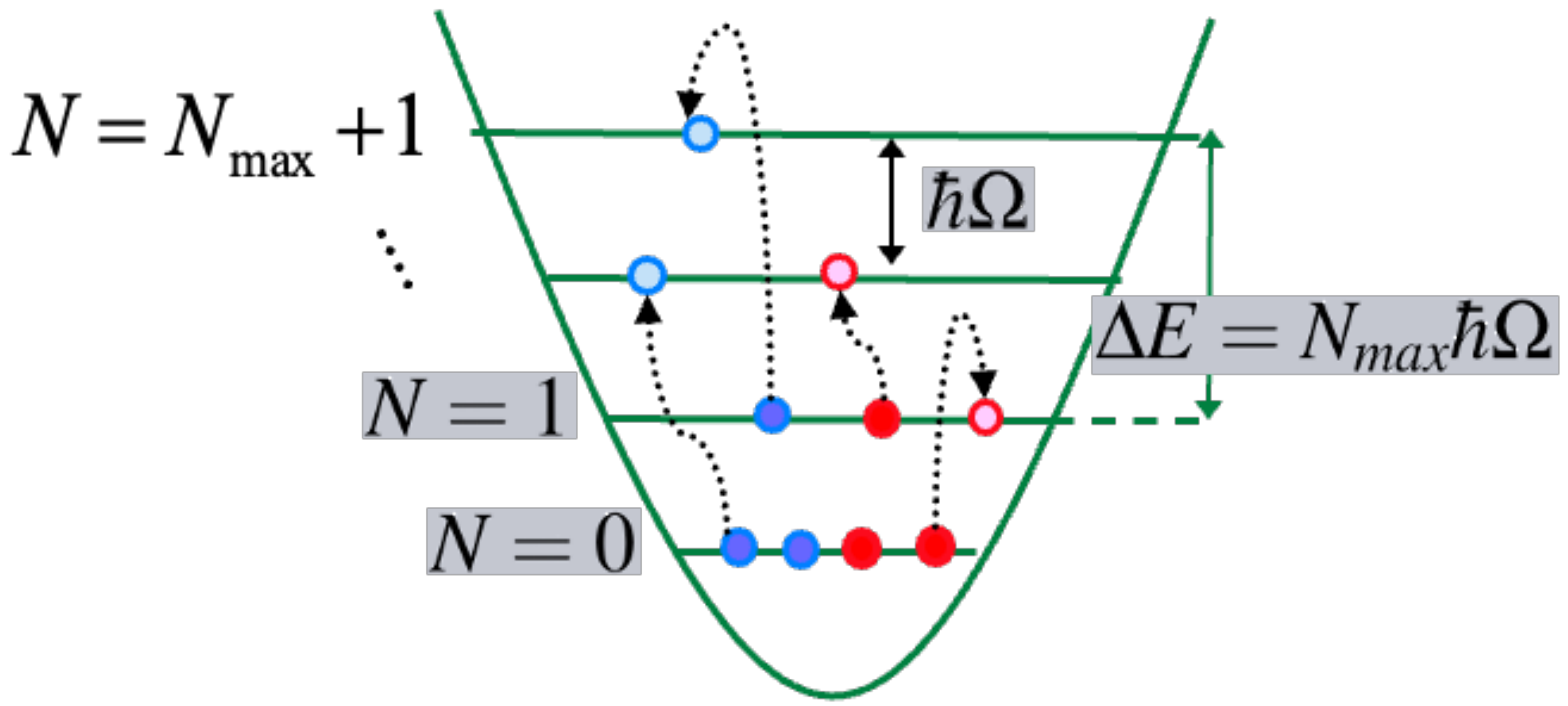
  ➡ **Estimate model space truncation uncertainty**
- Many-body approximation
  - No-Core Shell Model
  - More details in next section

  ➡ **Negligible many-body approximation uncertainty**

The No-Core Shell Model

Anti-symmetrized products of
many-body HO states



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Lanczos tridiagonalization algorithm [Lanczos (1950)]

- Initialization: normalized pivot $|\phi_1\rangle$
- Recursion: α_i, β_i and $|\phi_i\rangle$
  - $\beta_{i+1}|\phi_{i+1}\rangle = H|\phi_i\rangle - \alpha_i|\phi_i\rangle - \beta_i|\phi_{i-1}\rangle$
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- Output:
  - Lanczos basis and coefficients $\{|\phi_i\rangle, \alpha_i, \beta_i\}$ → H **in Lanczos basis**
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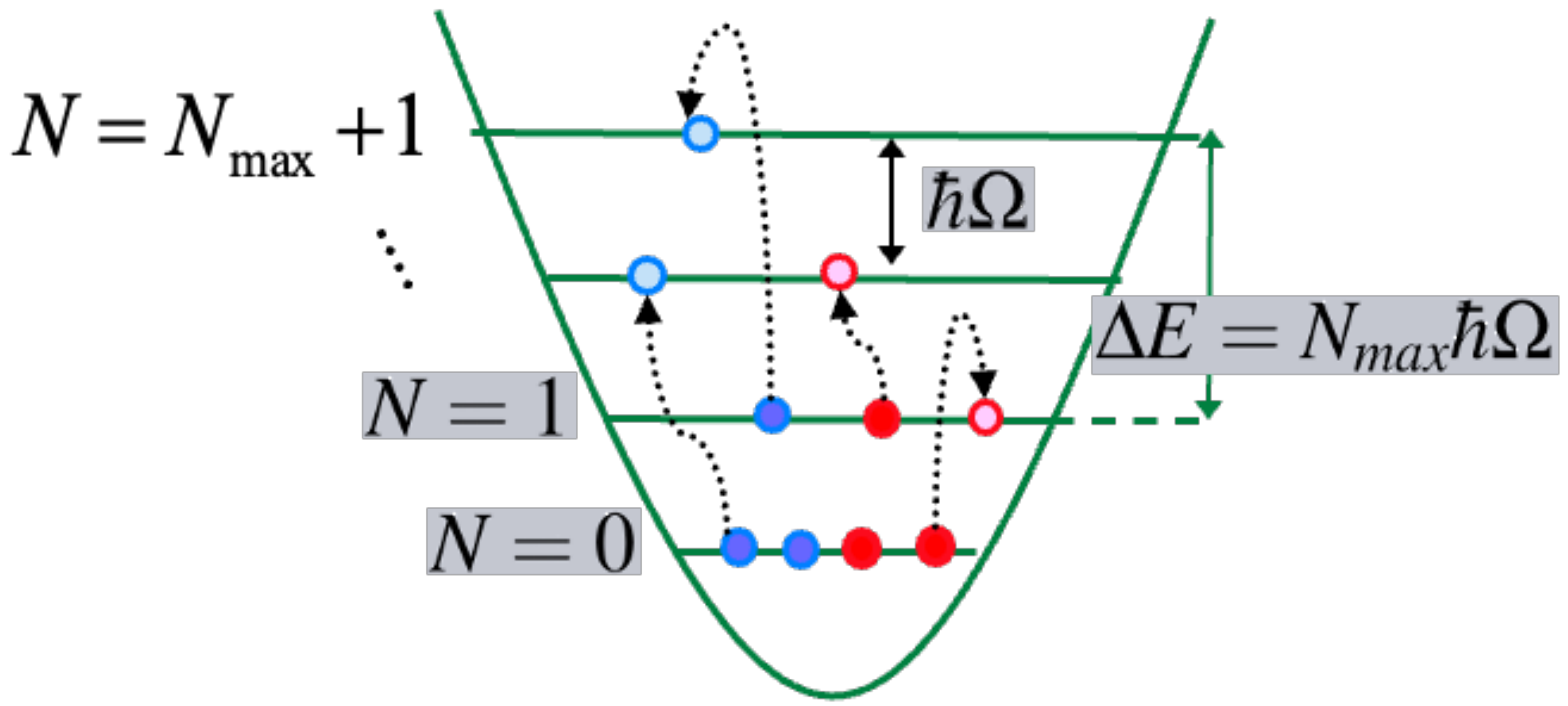

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Application to nuclear structure

- Efficient calculation of spectra
  - Selection rules sparsity ⇒ **Fast matrix-vector multiplication**
  - In practice: $N_L \sim 100 - 200$ is sufficient to converge low-lying states
  - Cost of diagonalization of the tridiagonal matrix is negligible

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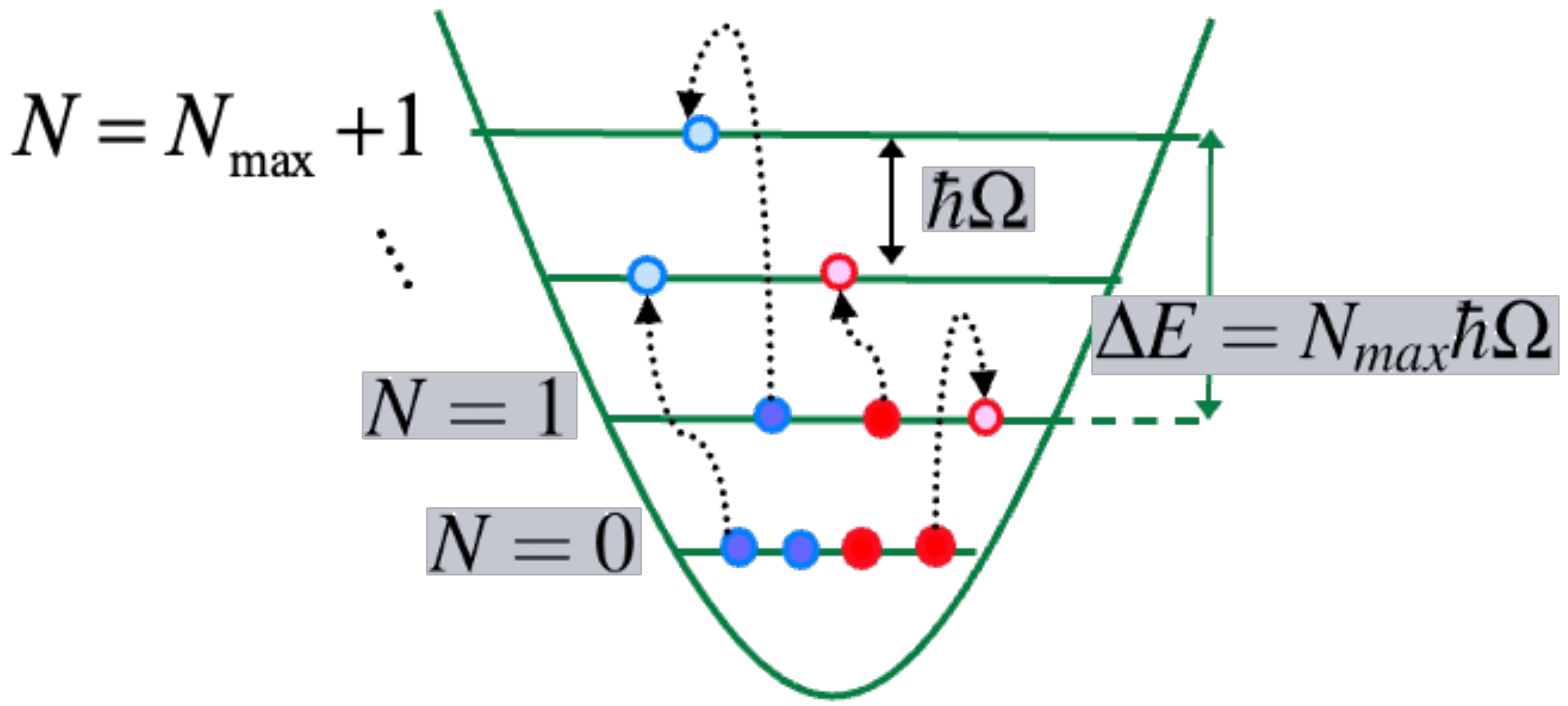

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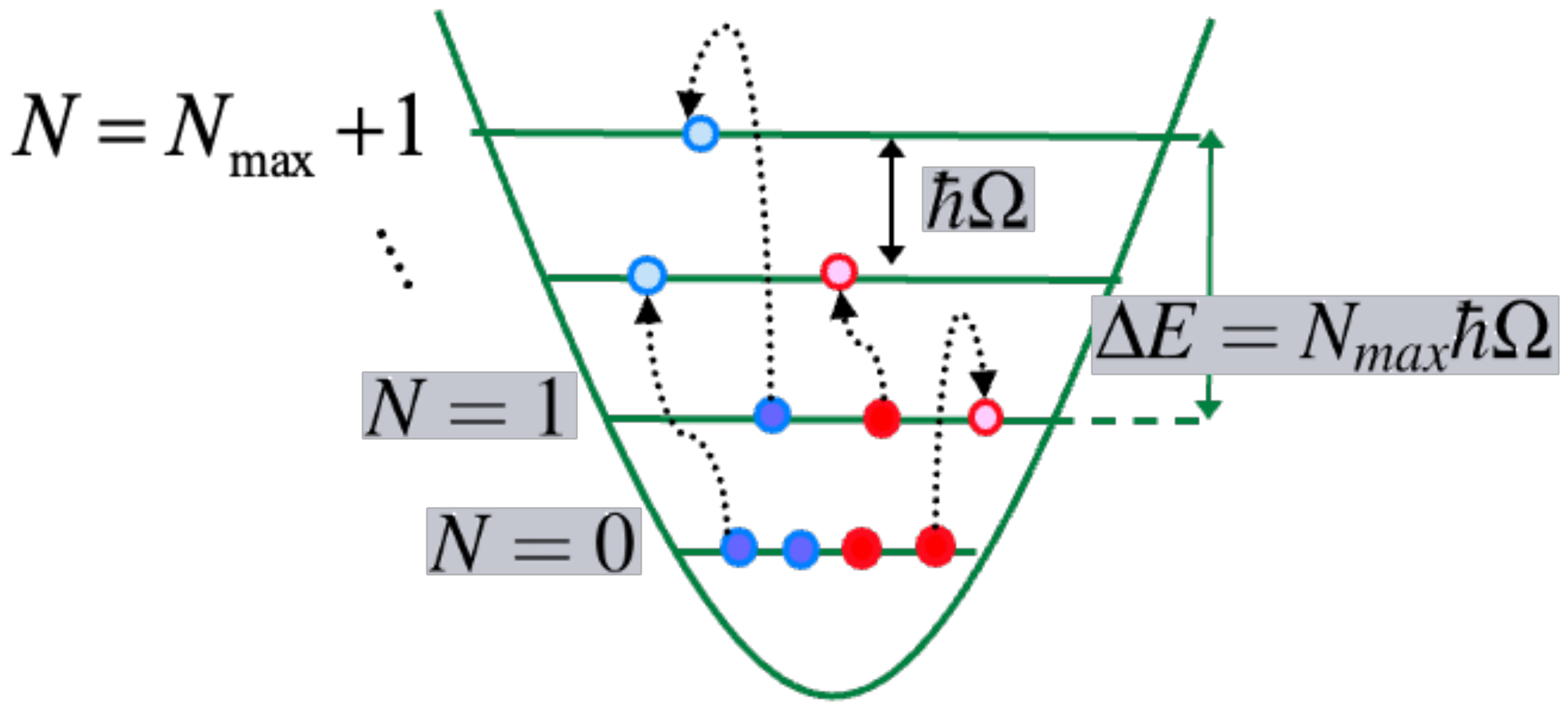

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Application to $^7\mathrm{Li}$

- Parameters of many-body calculation
  - $N_L = 200$ for $N_{\max} = 1$ to 9
- Results
  - Ground-state of $^7\mathrm{Li}$ $|\Psi\rangle$ ⇒ **Starting point for δ^A_{pol}**

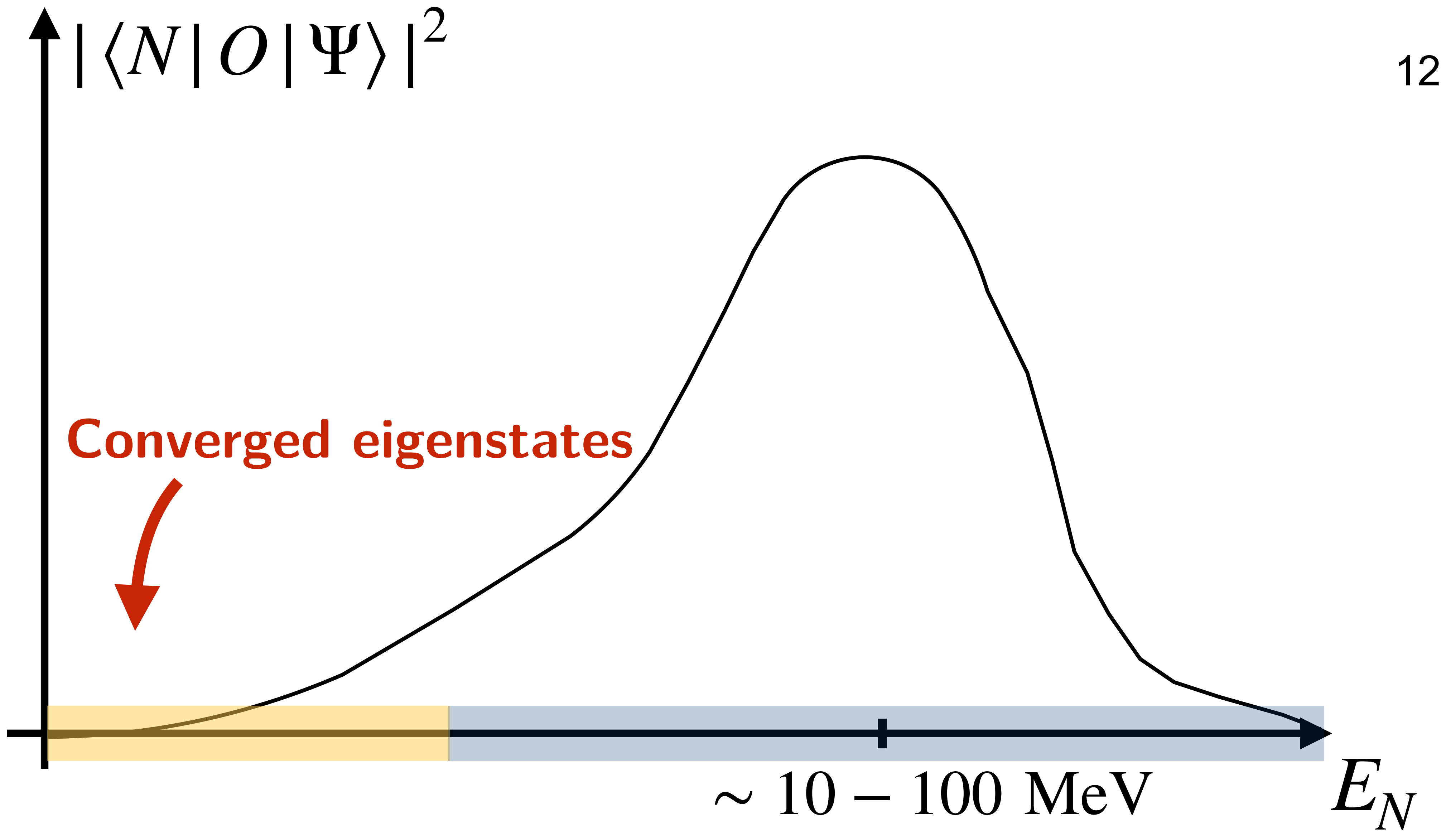
The Lanczos strength algorithm

Computing strength functions

- We need to compute for each eigenstate and operator:
  - Eigenvalues: E_N
  - Overlaps: $|\langle N|O|\Psi\rangle|^2$

  → **Too expansive to converge all of them !!**

- Lanczos strength algorithm
  - Variant of Lanczos: ensure convergence of **sum rules**



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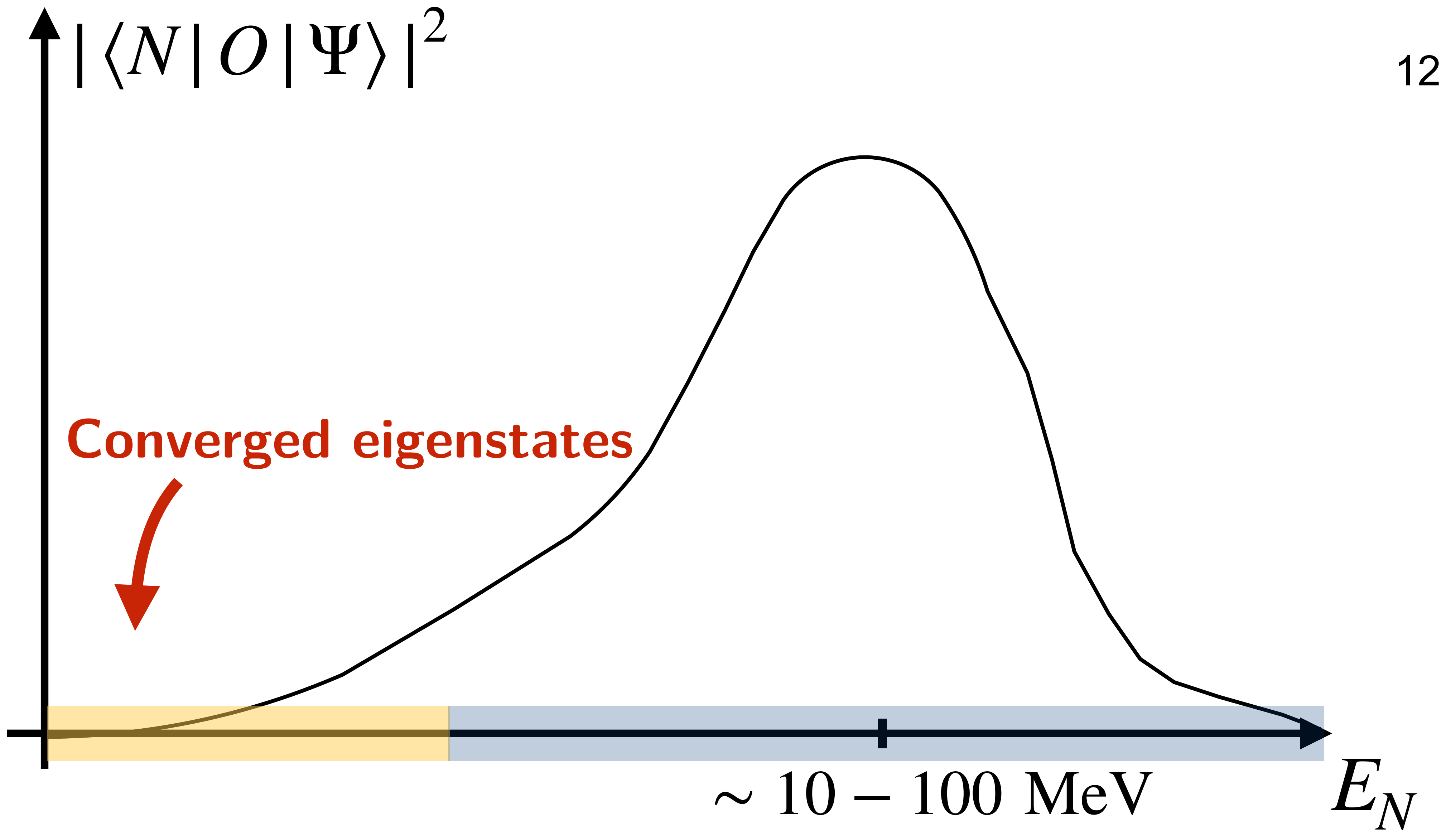

Sum rules convergence

- Convergence problem
  - Often the **strength is fragmented**
  - Only low-lying states converged in general
- Lanczos strength algorithm
  - Recover exactly $\int d\omega\, \omega^n\, S_O(\omega)$ for any $n \leq 2N_L$

  ➡ **Fast convergence of** $\int d\omega\, f(\omega) S_O(\omega)$ (if f $\sim P_{100}(\omega)$)

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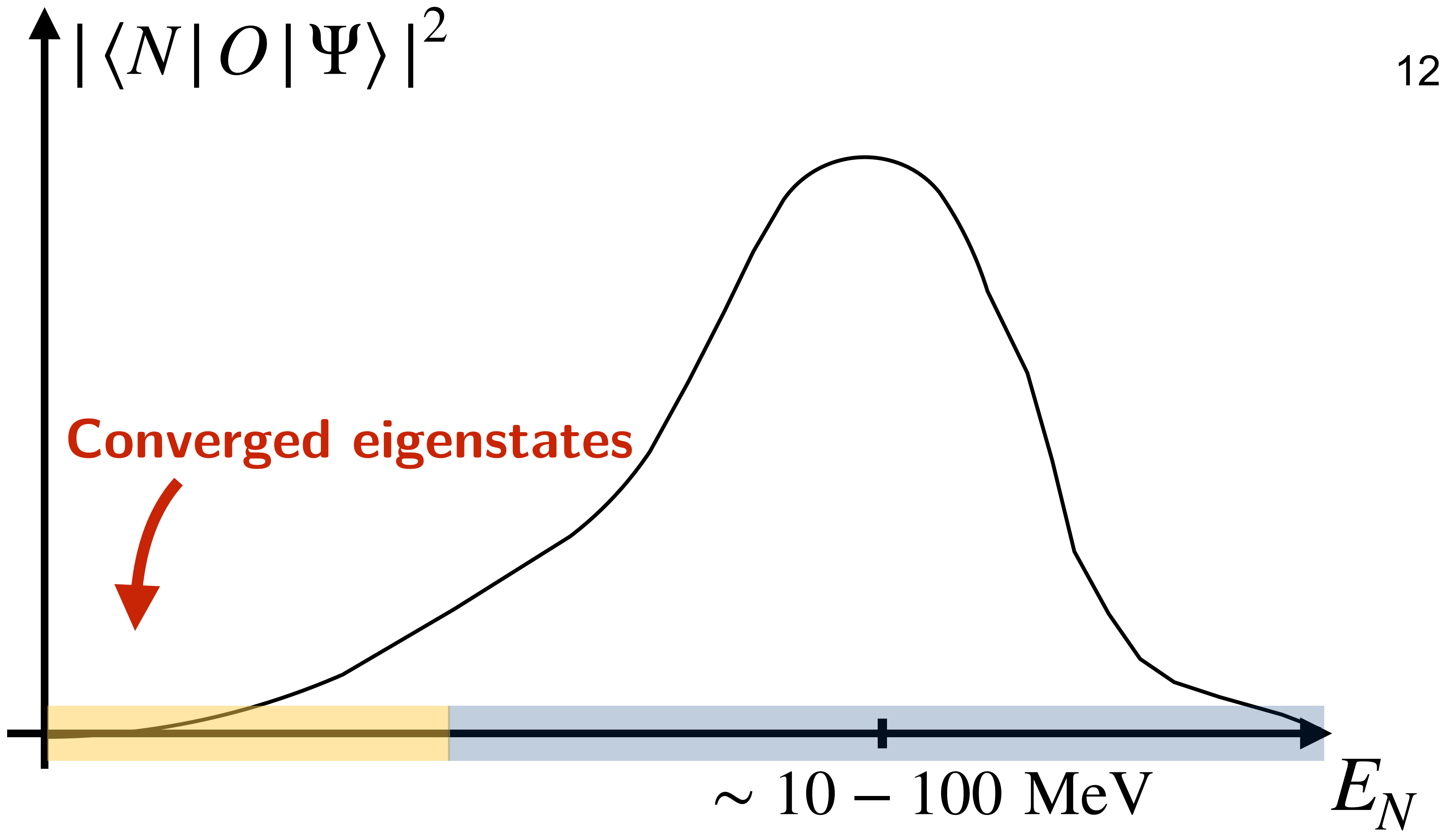

Main idea of the algorithm

- For each operator O
  - Compute $\frac{O|\Psi\rangle}{\sqrt{\langle\Psi|O^\dagger O|\Psi\rangle}}$ $\Rightarrow$ **Pivot** $|\phi_1'\rangle$ **for** 2^{nd} **Lanczos**
- Extract strength from orthonormality of Lanczos basis
  - $|\langle\Psi|O|N\rangle|^2 = \langle\Psi|O^\dagger O|\Psi\rangle \times |\langle\phi_1'|N\rangle|^2$

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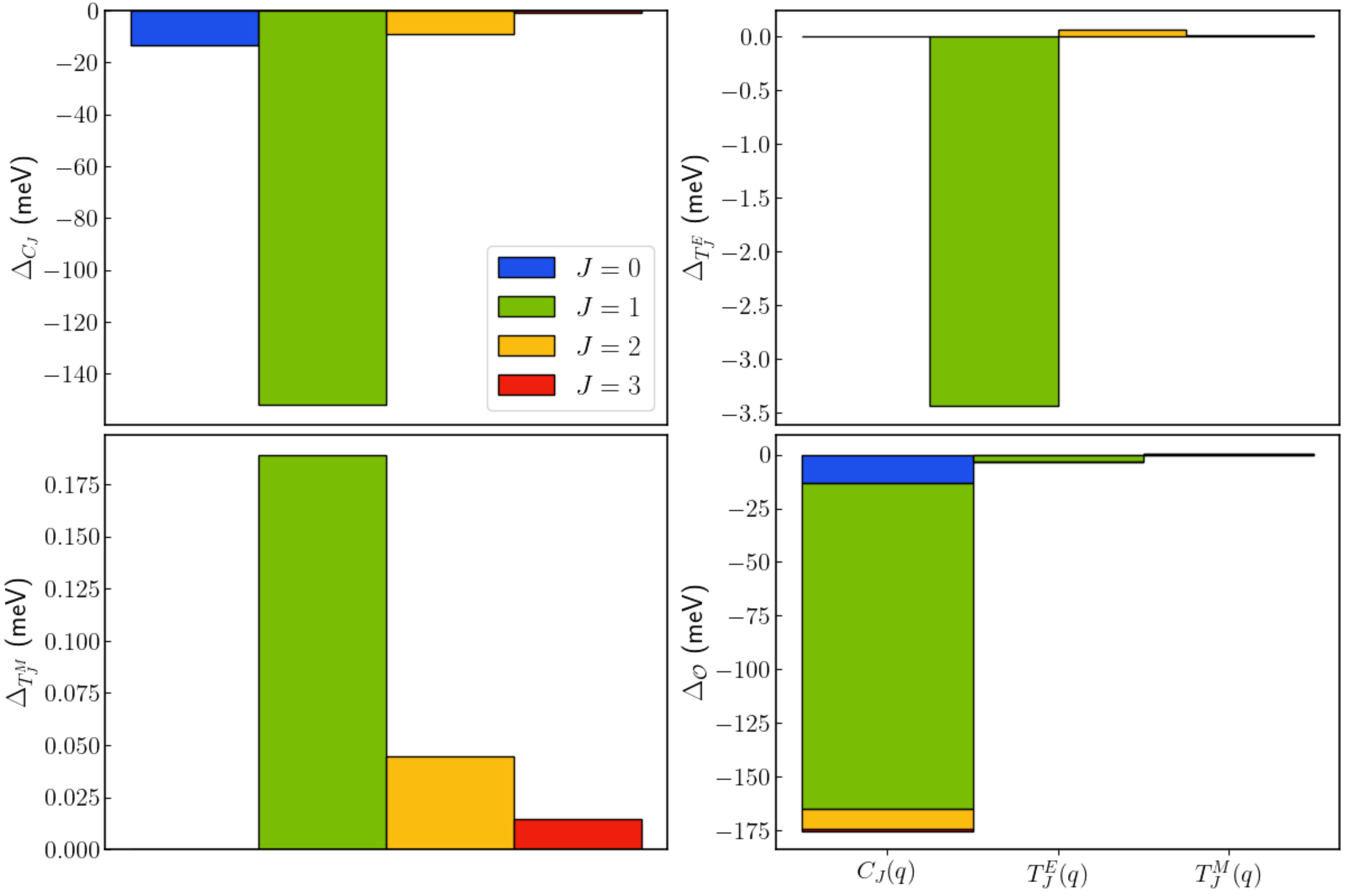
A first test case for N4LO-E7 and $N_{\max} = 7$

Numerical calculations

- $q_{\max} = 700$ MeV and $\Delta q = 10$ MeV
- 10 different operators for $J_{\max} = 3$

➡ **700 NCSM calculations at $N_{\max} = 7$**

A first test case for N4LO-E7 and $N_{\max} = 7$



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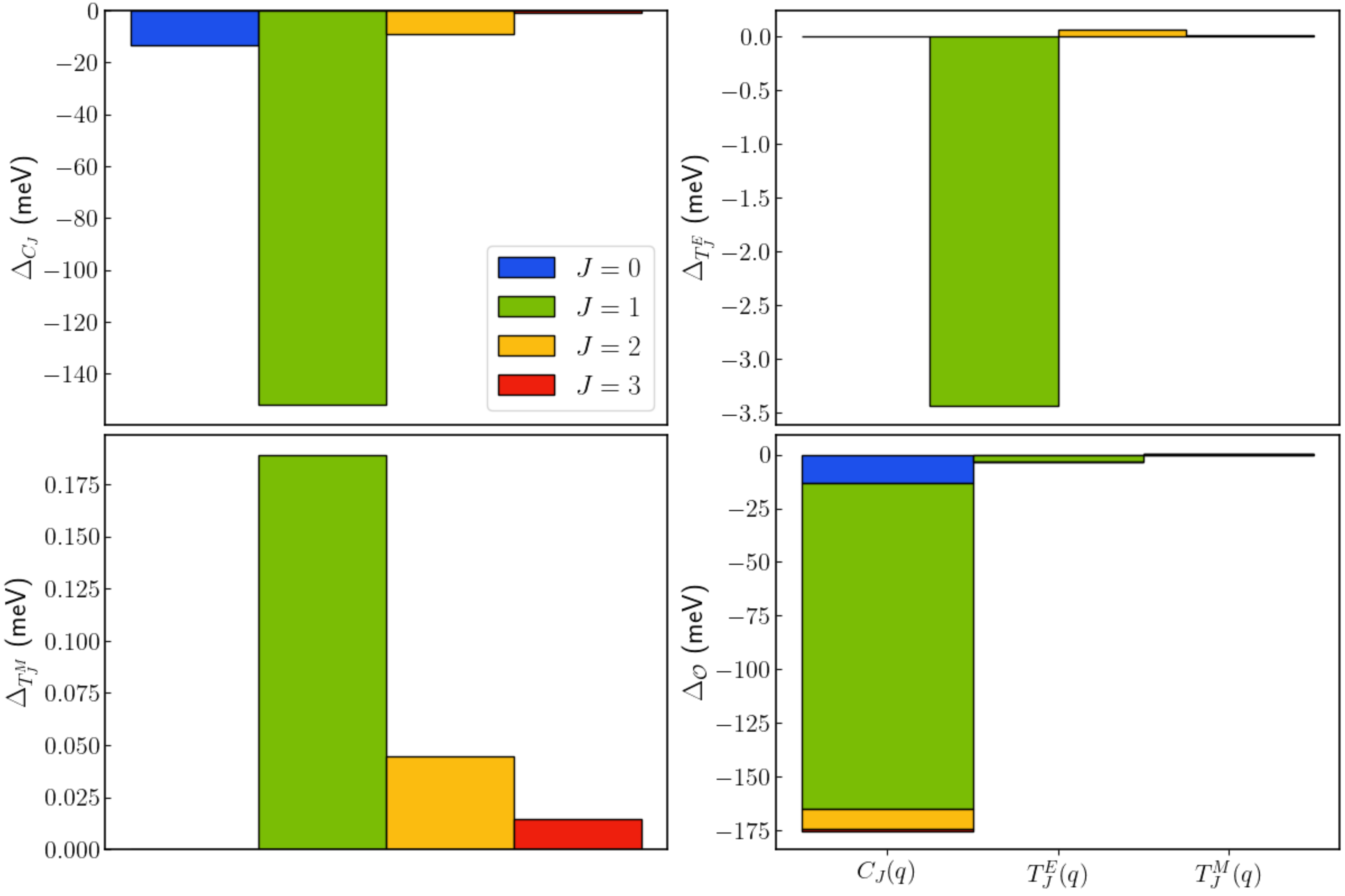

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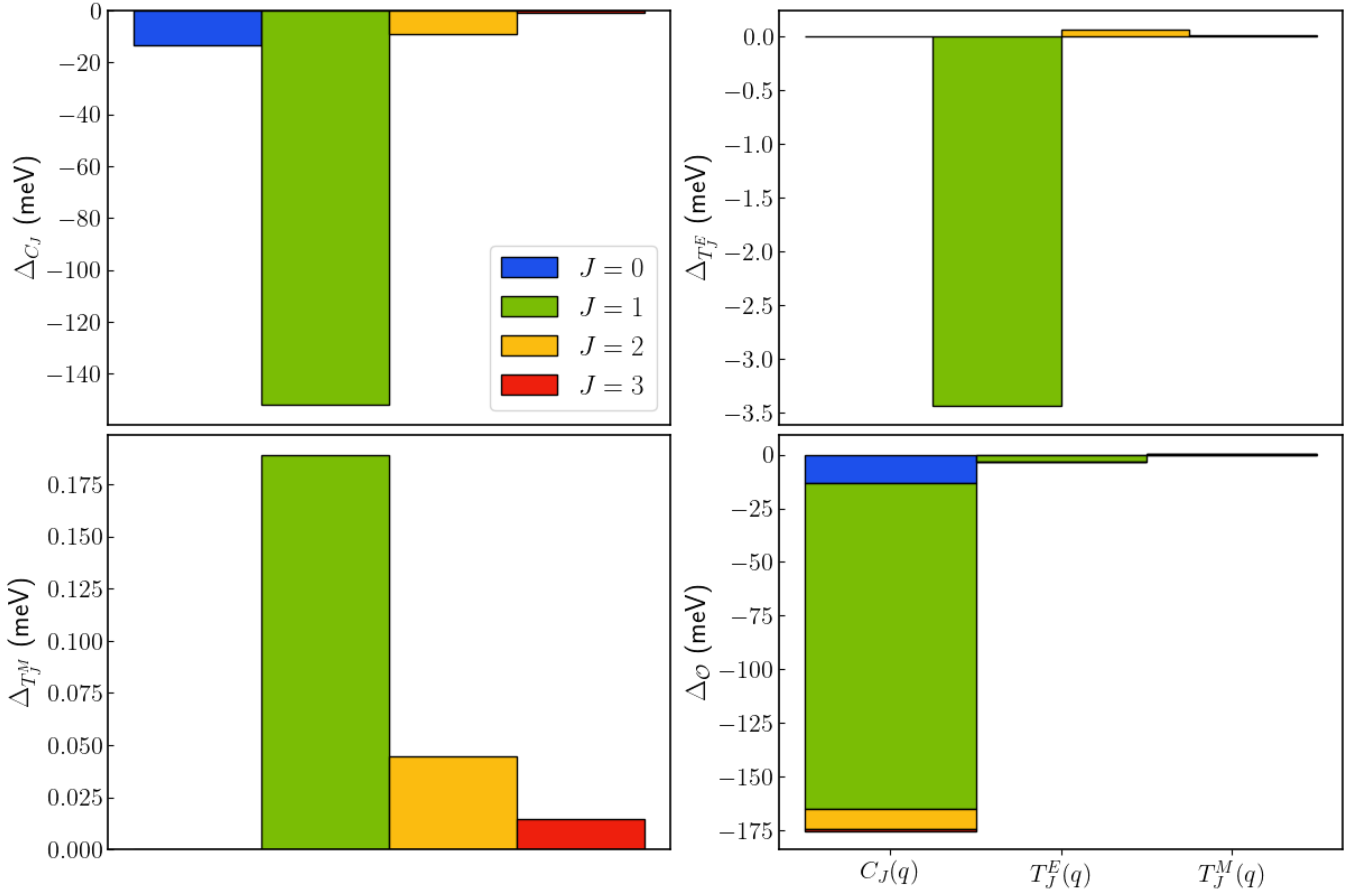

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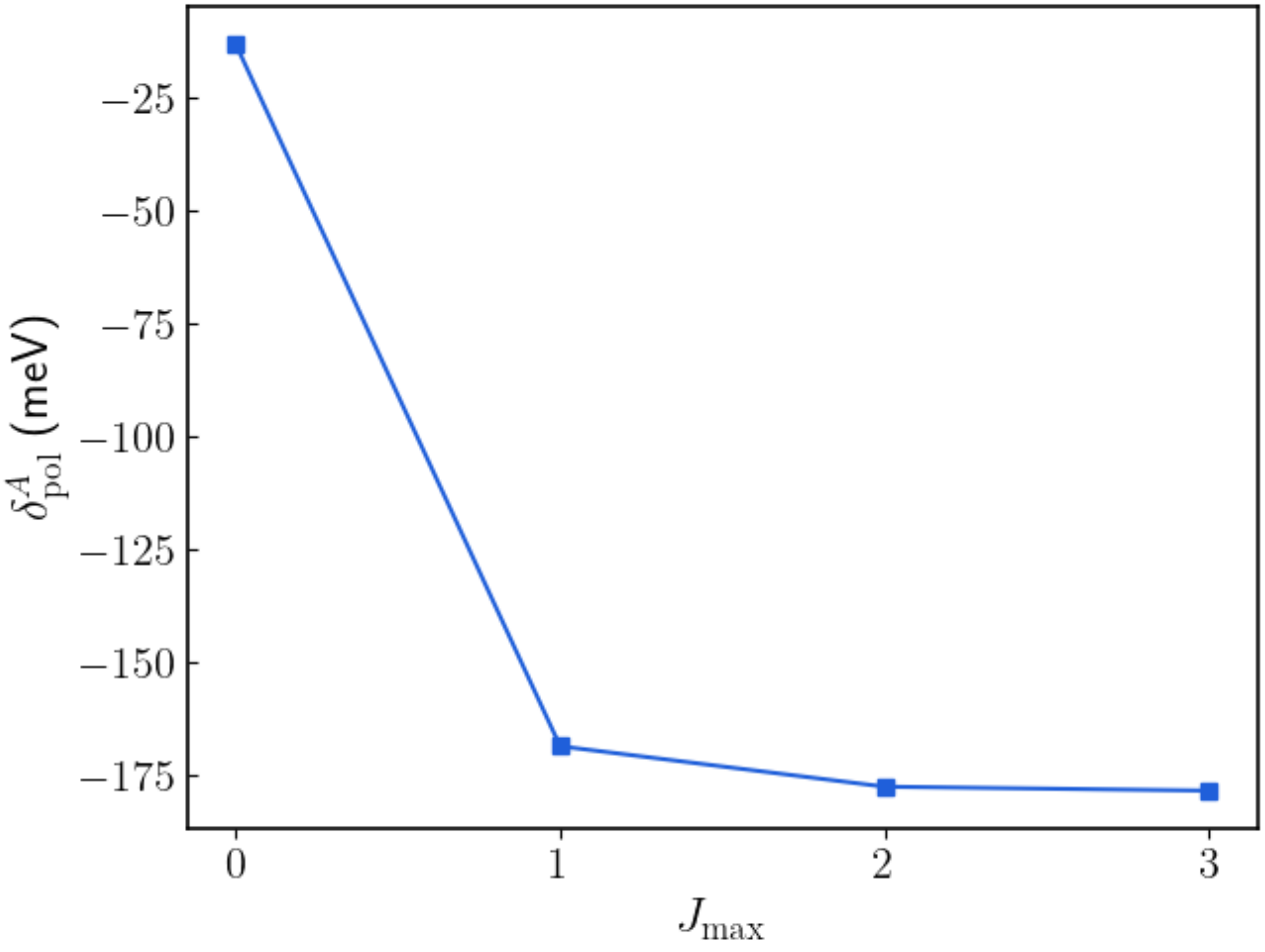
- Negligible contributions
  - TM is negligible for any J
  - TE is relevant only for $J = 1$

➡ **Only half the operators are relevant**

Checking convergence in $J_{\max}$



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- Here shown for $N_{\max} = 7$ and N4LO-E7
- All other cases are similar

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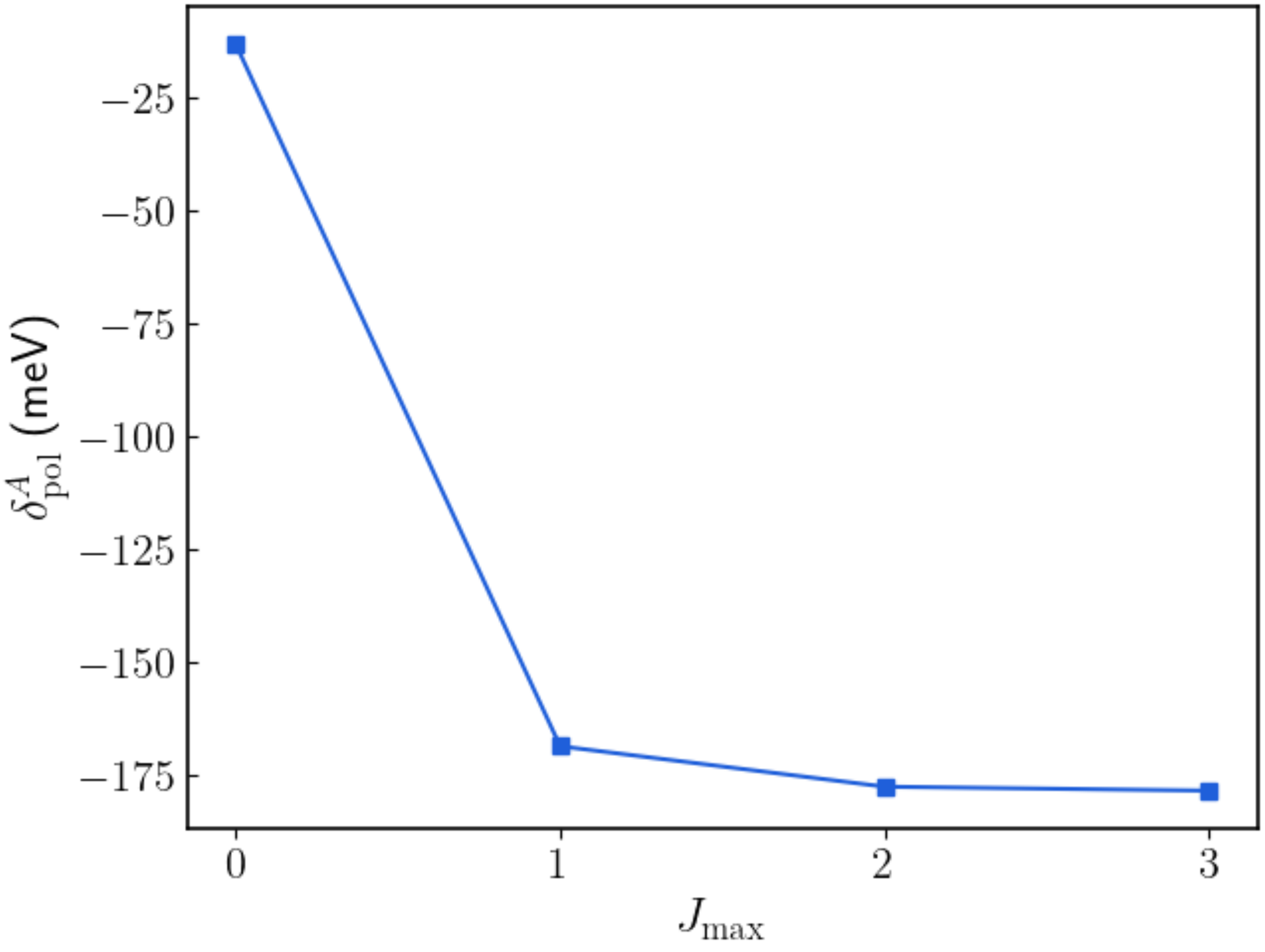

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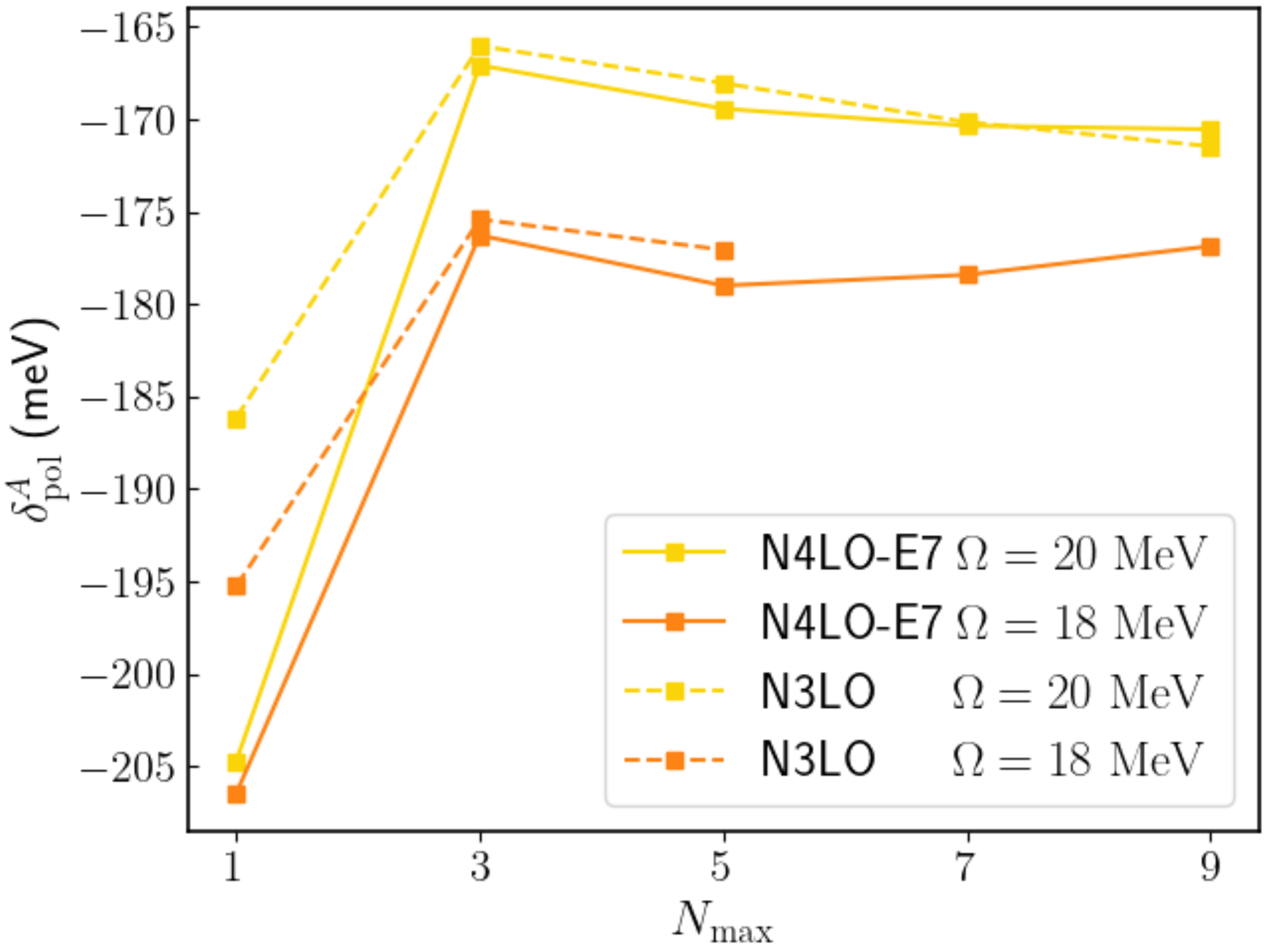
➡ **Fast exponential convergence**

$$\epsilon_{J_{\max}} \lesssim 0.1 \text{ meV}$$

Multipole truncation ⇒ Negligible uncertainty

Dependence on $(\Omega, N_{\mathrm{max}})$ and the interaction



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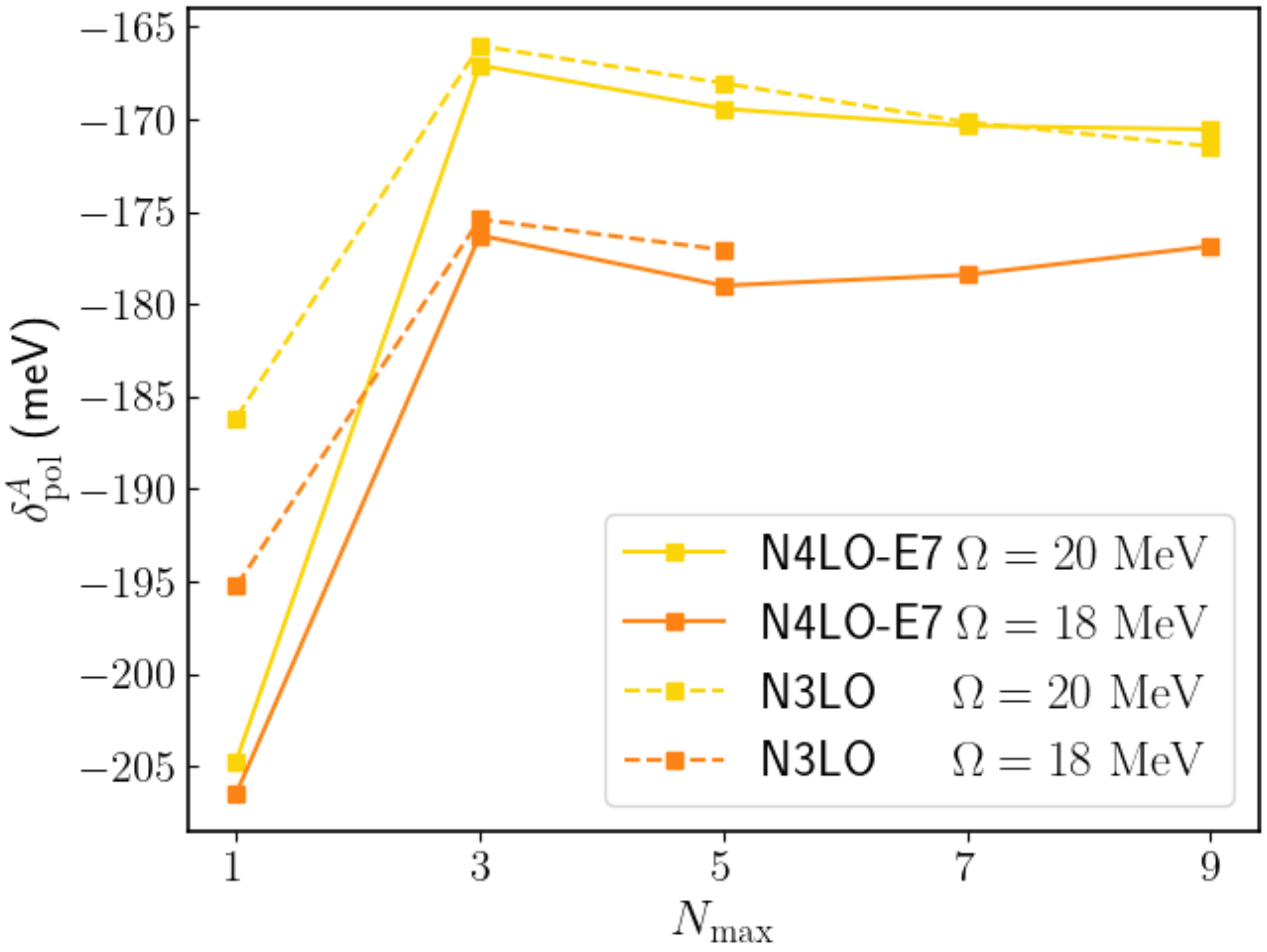

Numerical results

- Model-space dependence
  - Optimal frequency around 20 MeV
  - Run calculations for $\Omega = 18,\ 20$ MeV
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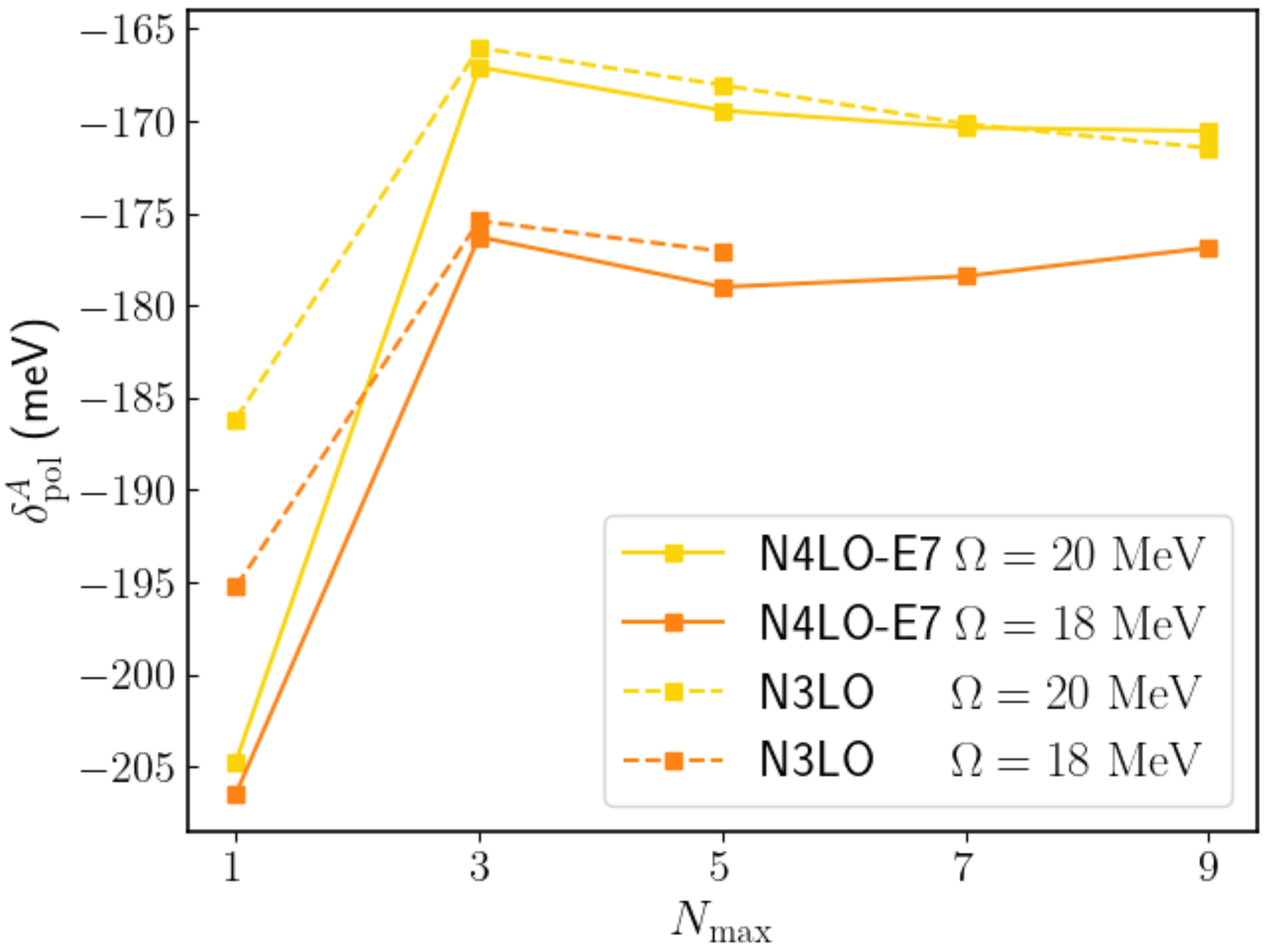

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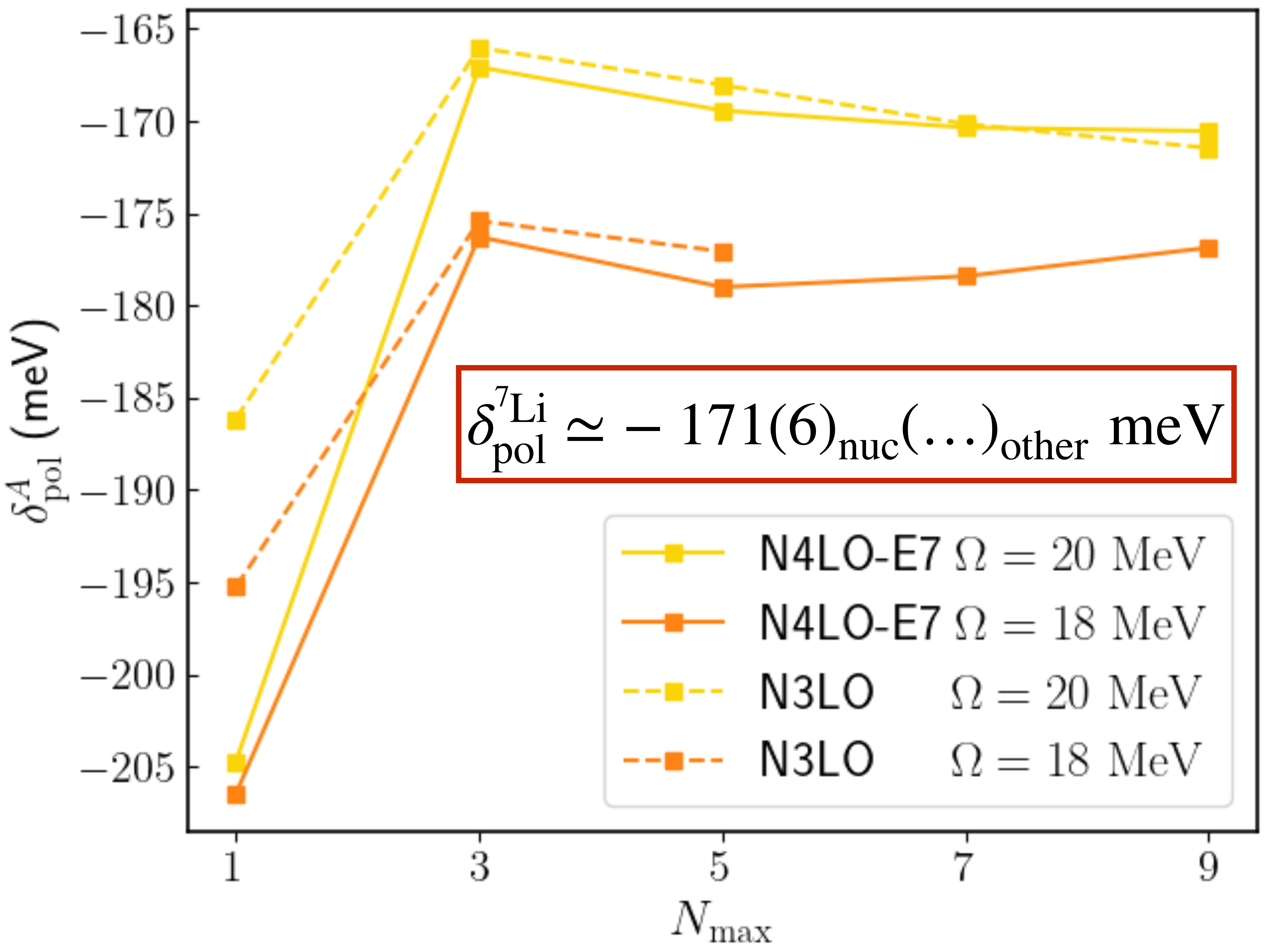

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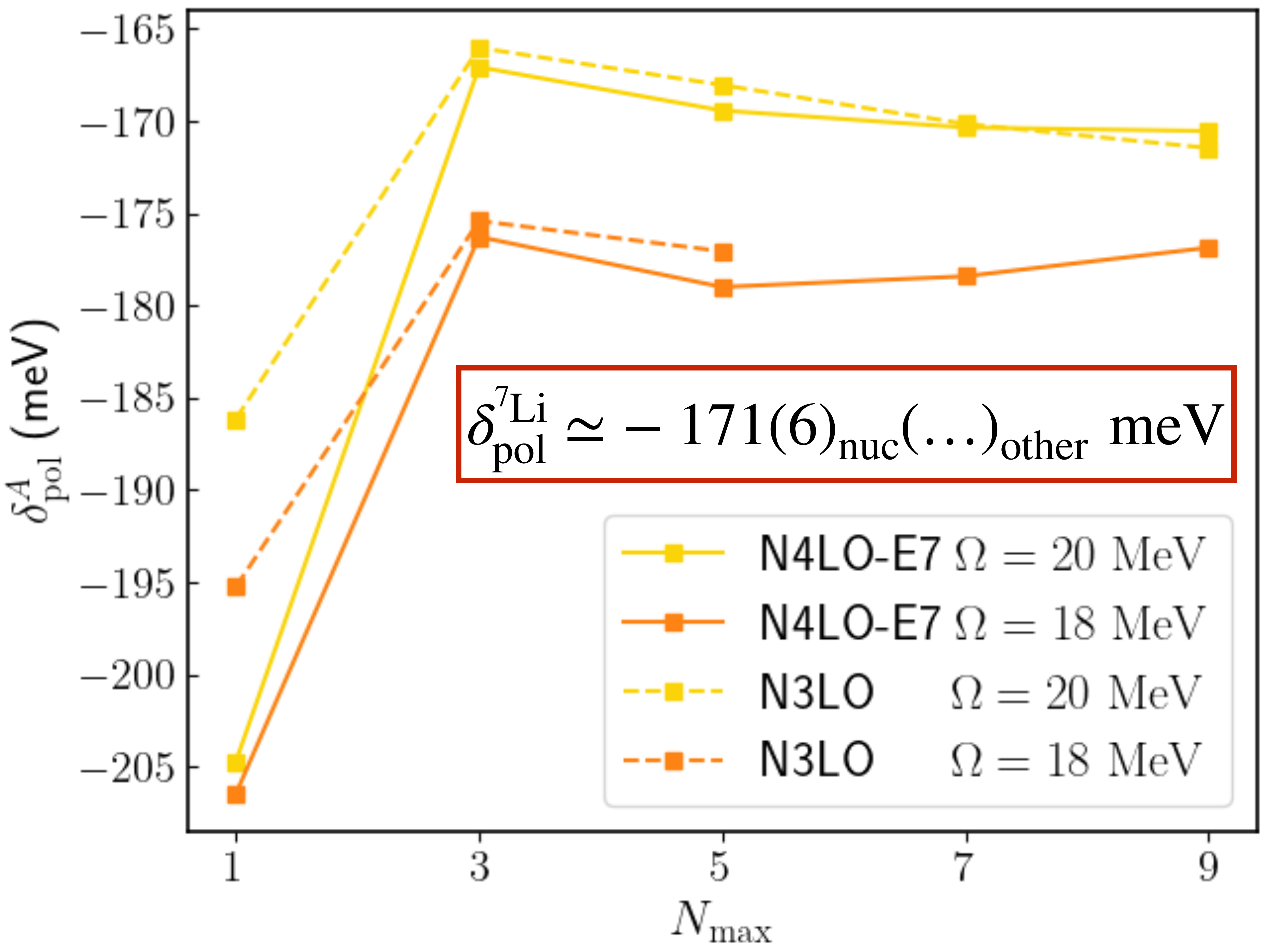

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A 10 meV precision for nuclear structure corrections seems doable in the near future!

Conclusion

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- Muonic atoms: a precision probe for nuclear physics
  - Radii extraction: reference point + isotope-shift
  - Precise reference point: muonic atoms
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Outlook

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  - Refining uncertainty quantification
  - Elastic component: δ^A_{el} with NCSMC
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- Towards better controlling theoretical uncertainty
  - Shifting from pheno towards EFT approach
  - EFT based on **potential-NRQED** for $Z > 1$

[Peset et al., EPJA (2015)]

Thank you
Merci

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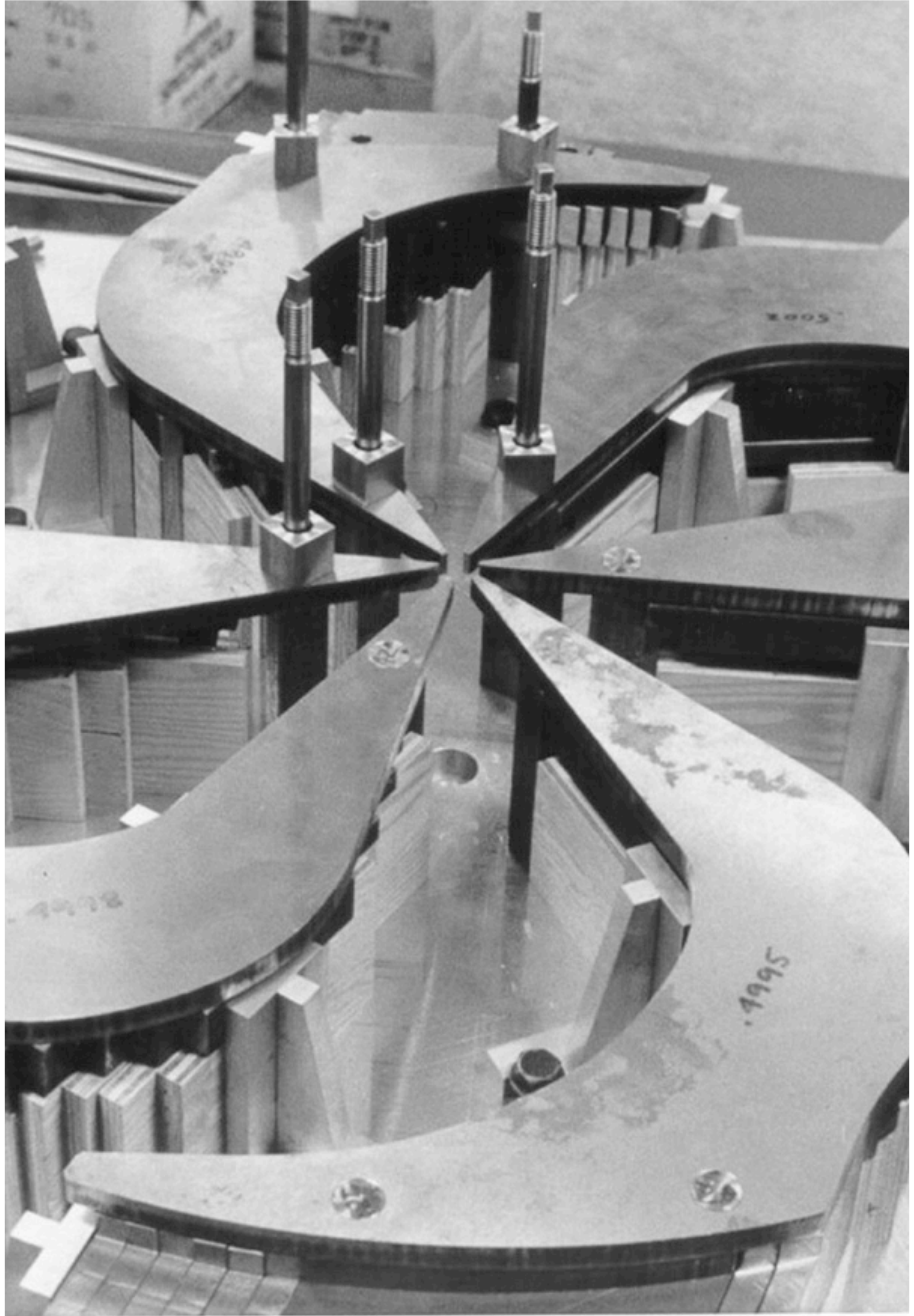
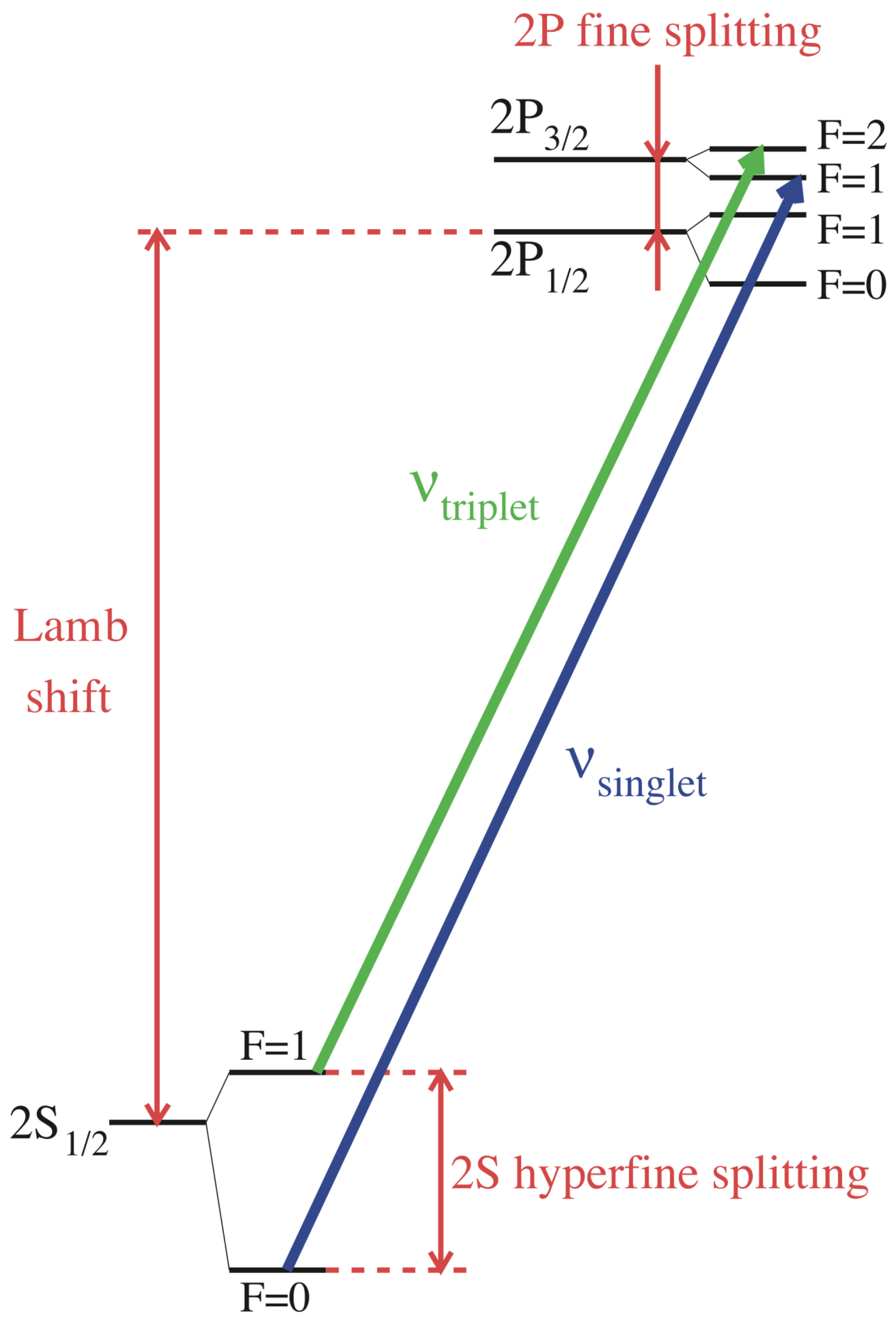


Discovery,
accelerated

Backup slides

The muonic Lamb shift as a precision probe



[Antognini et al, Science (2013)]

The muonic Lamb shift as a precision probe

A key probe to develop the Standard Model…

~1900 **Observation of atomic spectra**

↓ Discrete energy levels
Quantum theory

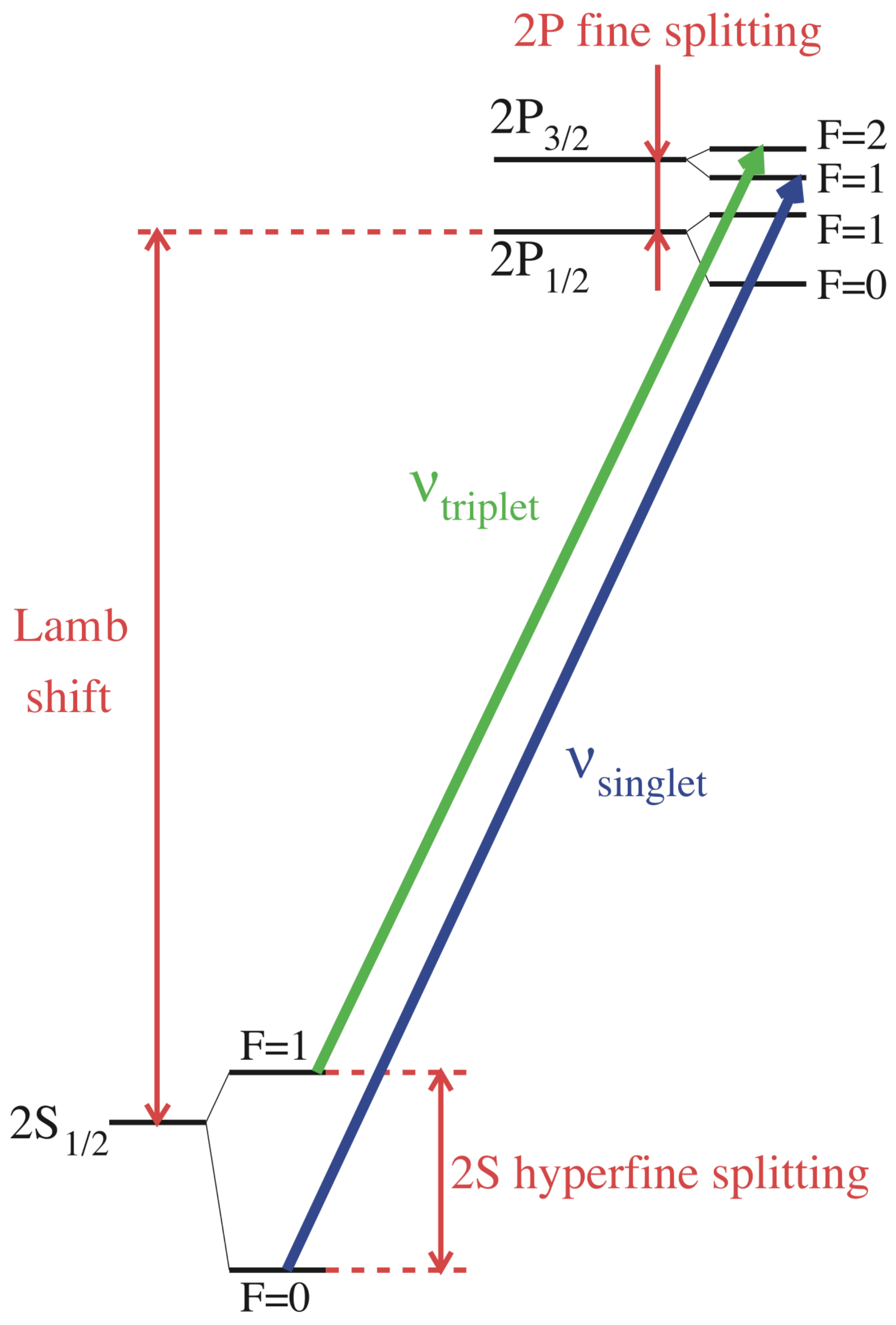
~1947 **Observation of Lamb shift between $2S_{1/2} - 2P_{1/2}$ states**

↓ Vacuum polarization effect
Renormalization theory

~1965 **Quantum Electrodynamics**

↓ Non-abelian generalization

~1970 **Standard Model of particle physics**



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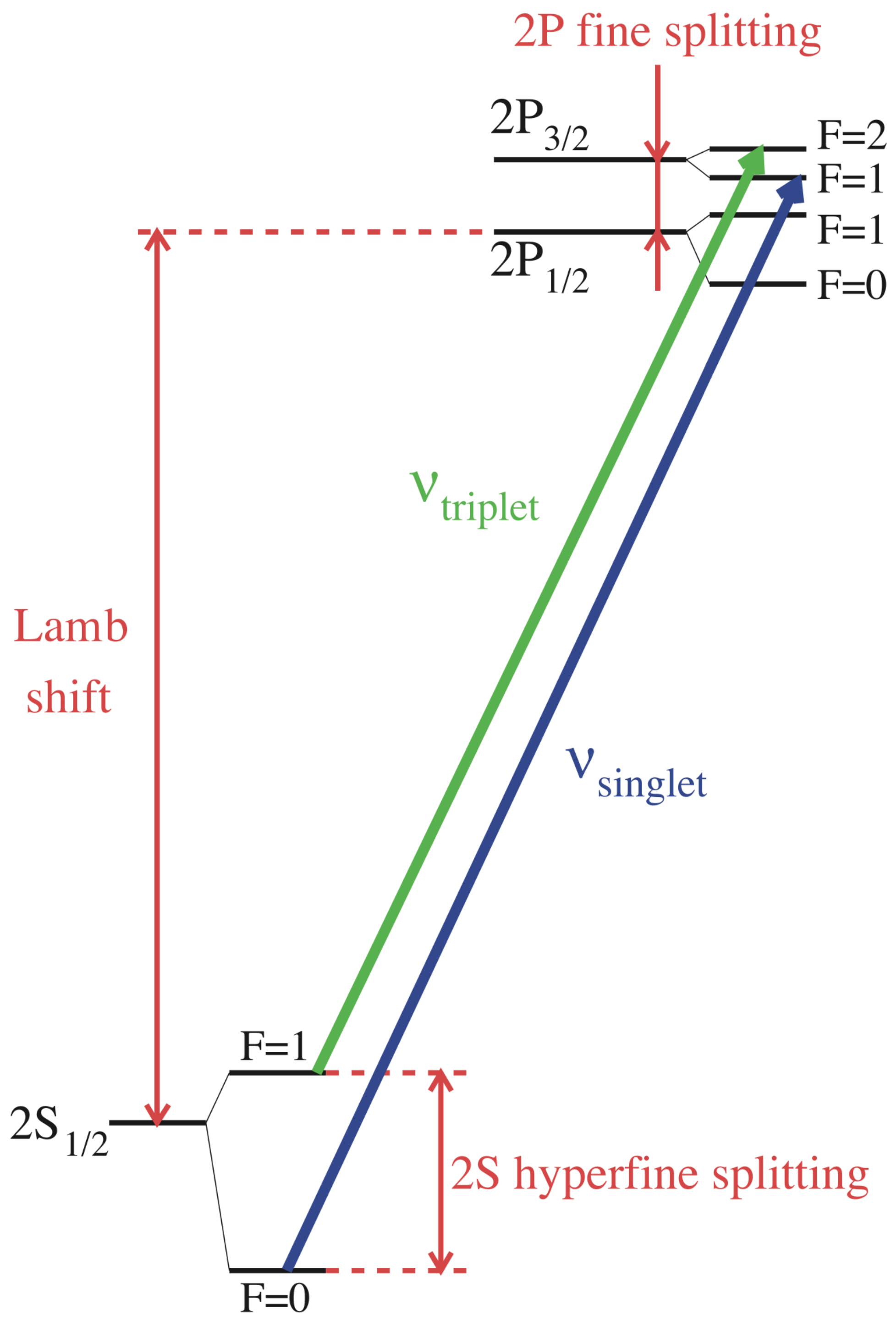

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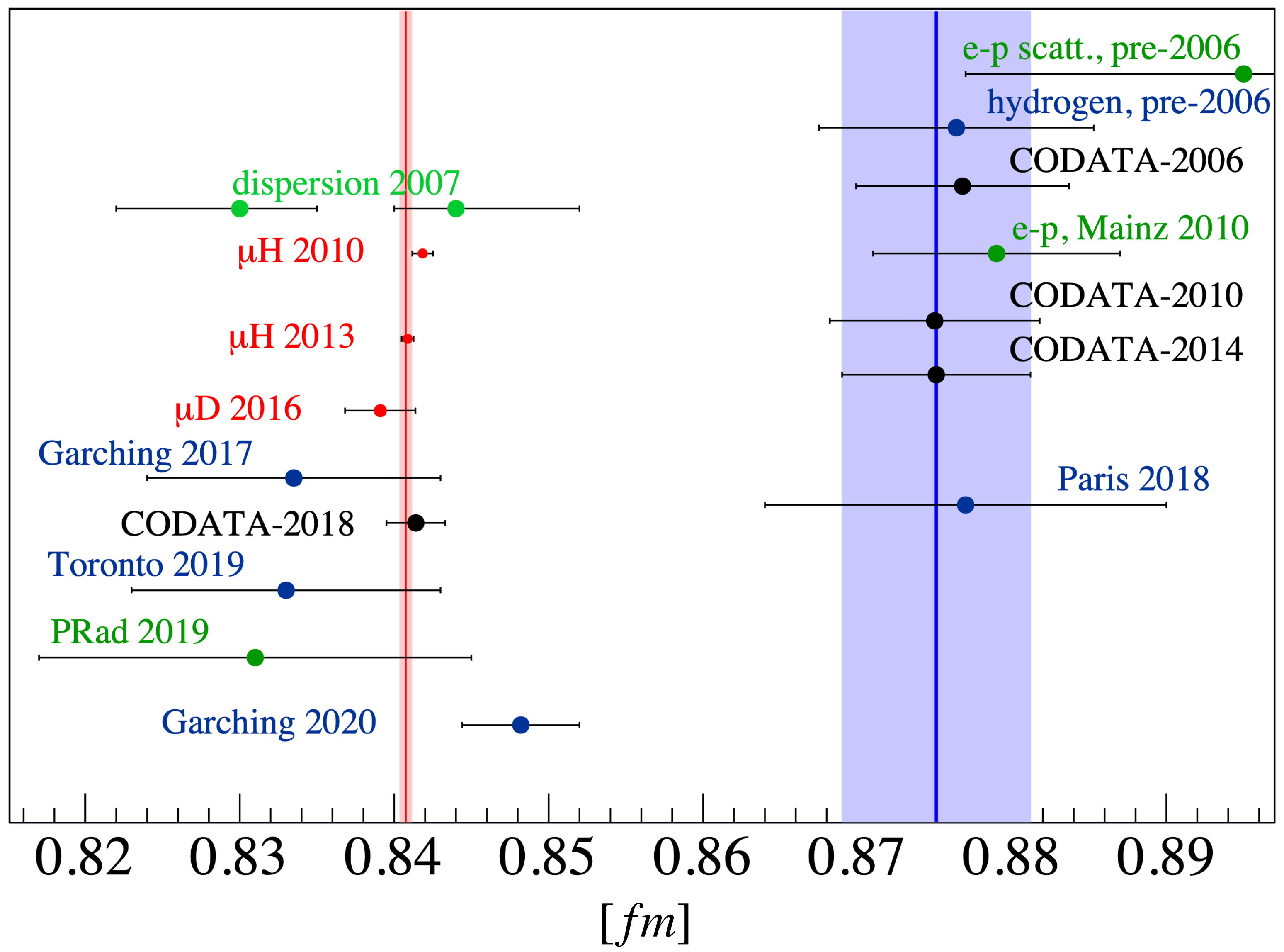
… and pushing the precision frontier further



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  [CODATA 2018]

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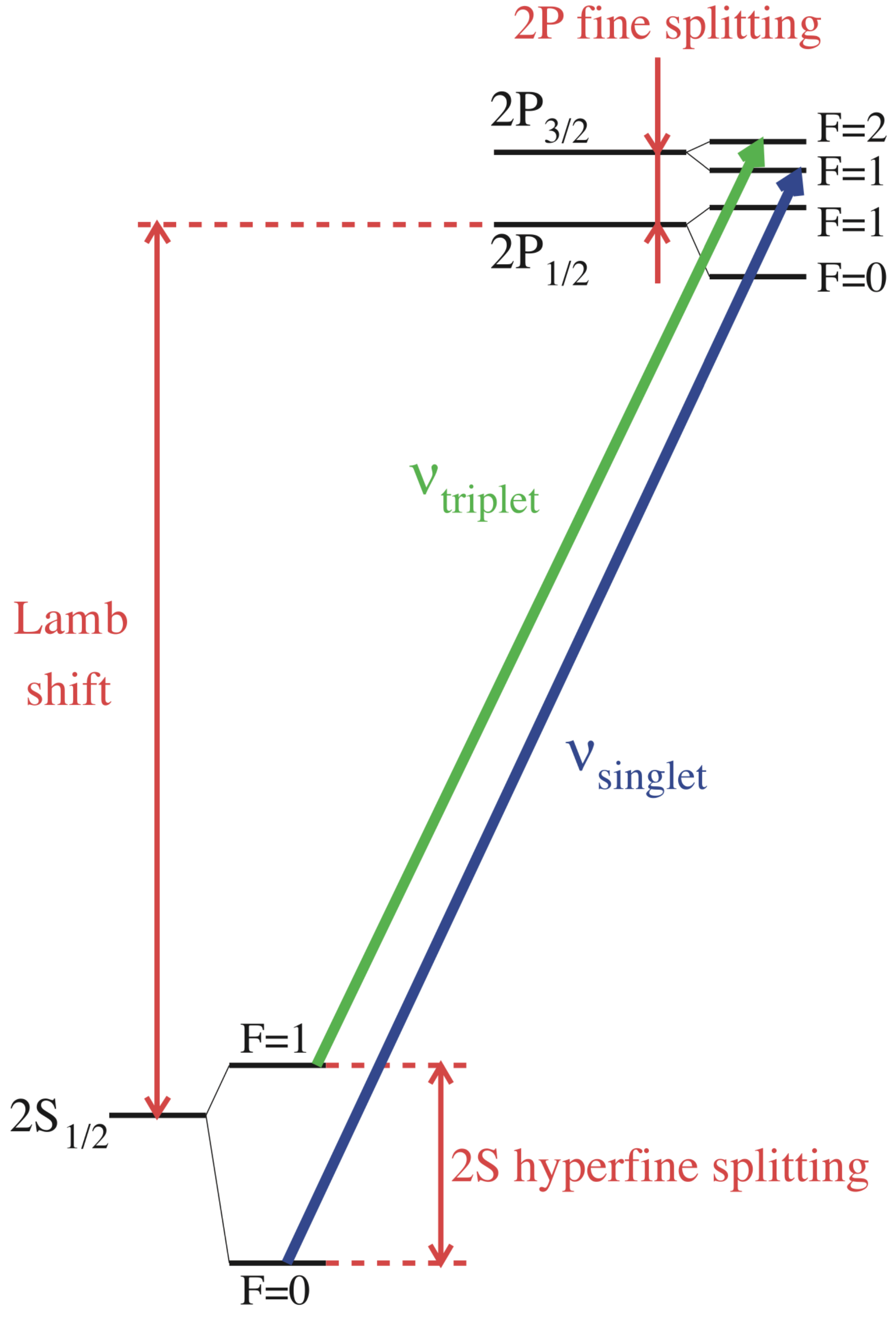

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And much more !!

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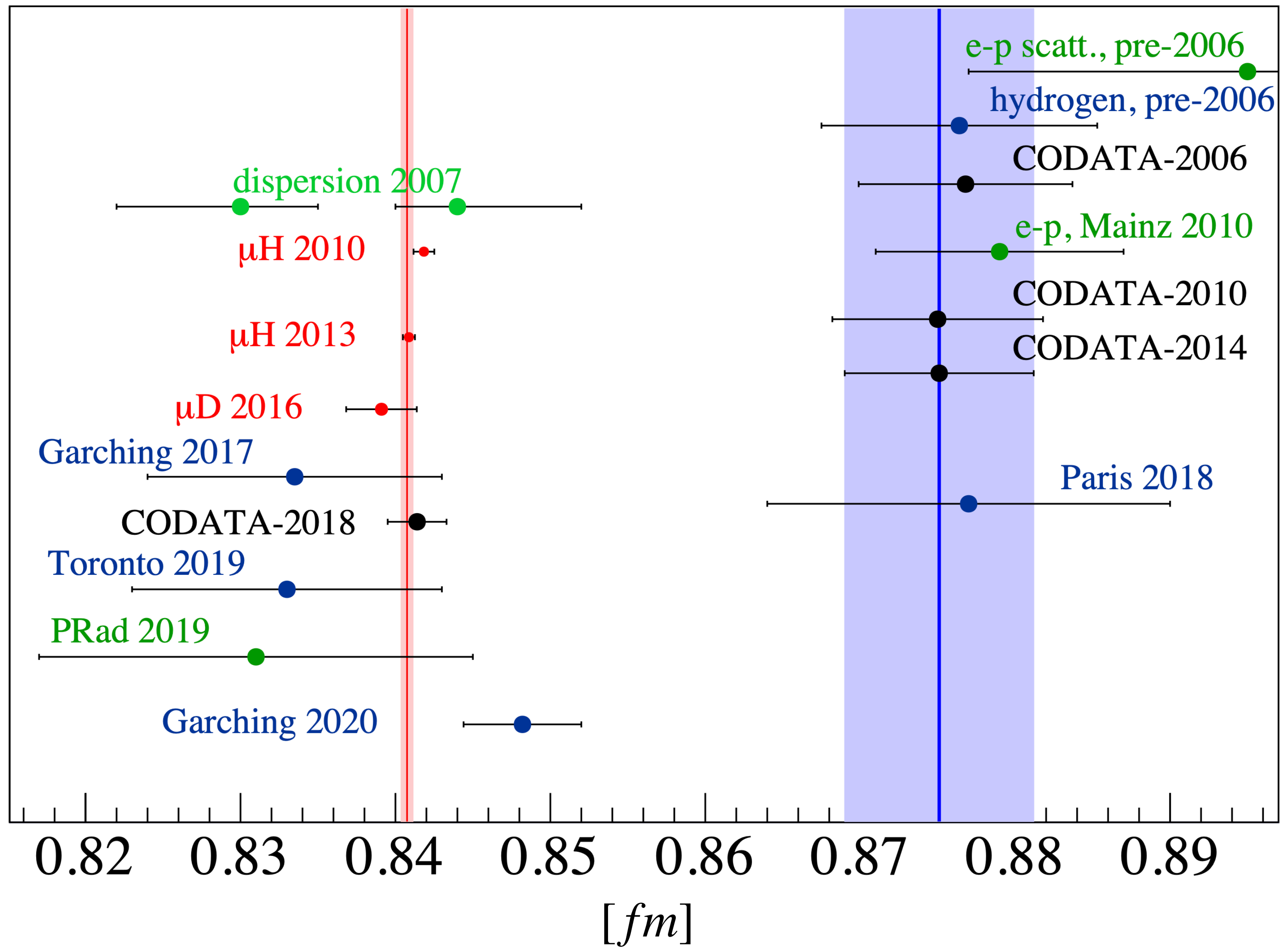

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- Correction to account for non-point like nucleus
  - Similar approach as pure QED contributions
  - Multipole expansion of charge distribution
  - ➡ Main contributions $\propto r_c^2$

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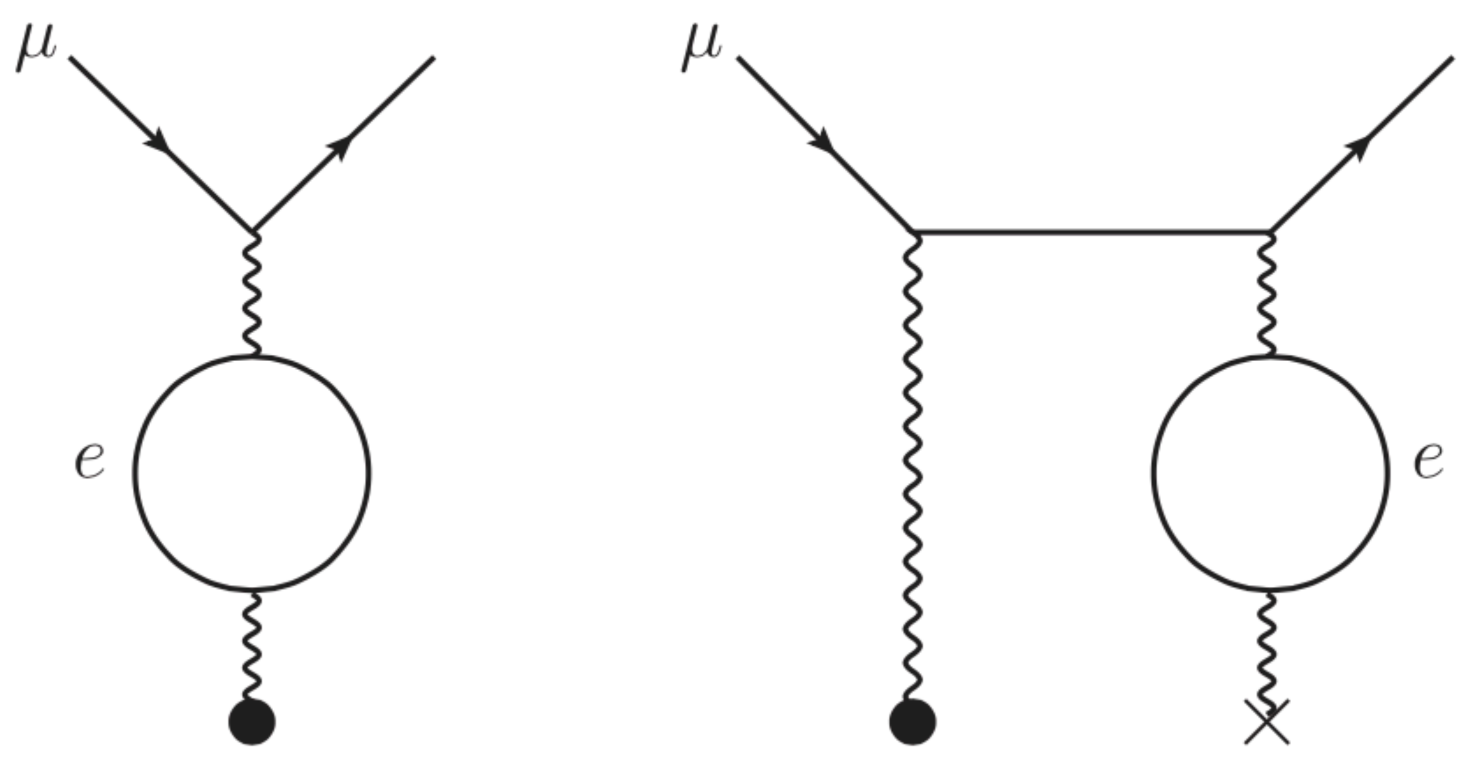
Examples with electron vacuum polarization



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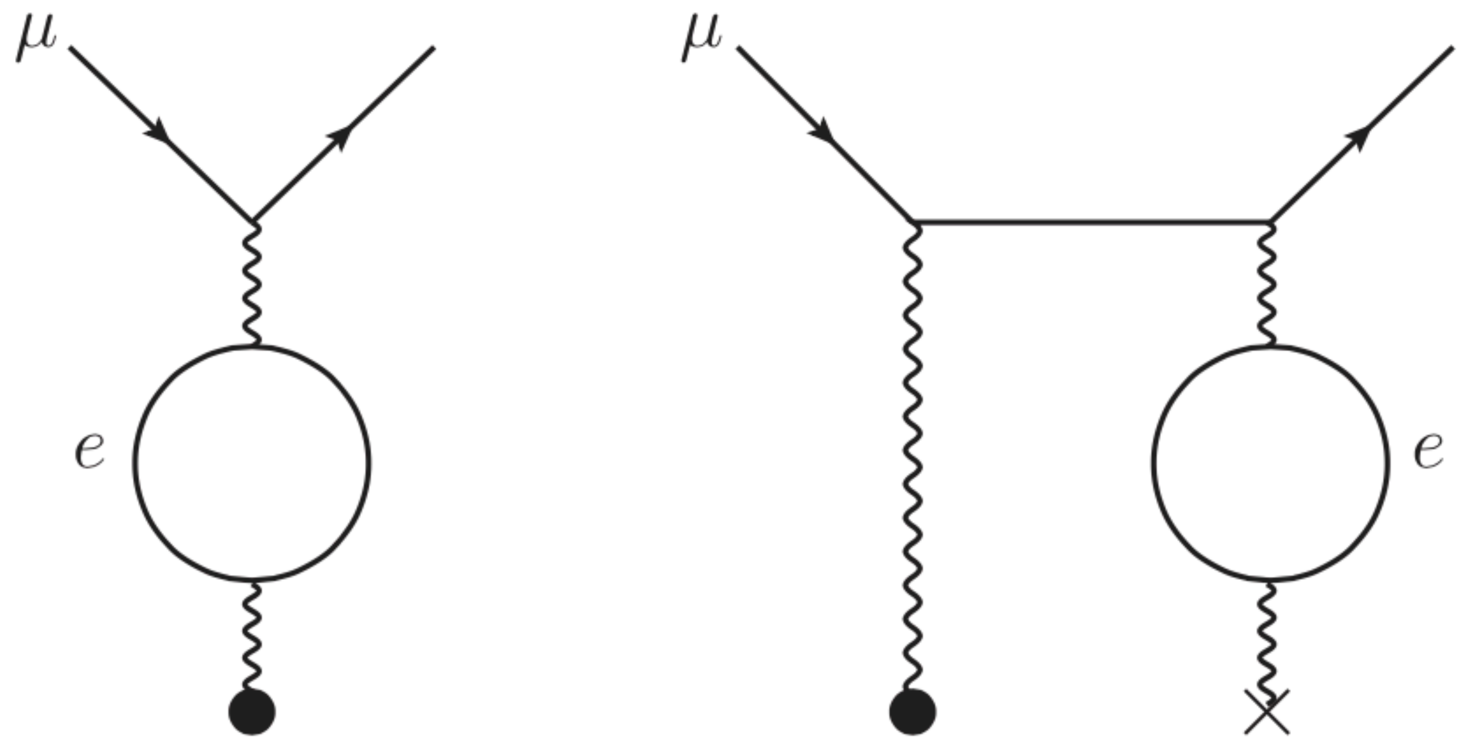

$\Rightarrow \quad \mathscr{C} r_c^2$ term in δ_{LS}

[Pachucki et al. Review of Modern Physics (2024)]

Section	Order	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
IV.A	$(Z\alpha)^4$	r_C^2	$-5.197\,5 r_p^2$	$-6.073\,2 r_d^2$	$-102.523 r_h^2$	$-105.322 r_\alpha^2$
IV.B	$\alpha(Z\alpha)^4$	eVP$^{(1)}$ with r_C^2	$-0.028\,2 r_p^2$	$-0.034\,0 r_d^2$	$-0.851 r_h^2$	$-0.878 r_\alpha^2$
IV.C	$\alpha^2(Z\alpha)^4$	eVP$^{(2)}$ with r_C^2	$-0.000\,2 r_p^2$	$-0.000\,2 r_d^2$	$-0.009(1) r_h^2$	$-0.009(1) r_\alpha^2$

Nuclear structure dependent corrections

Nuclear structure effects

- Corrections accounting for non static effects
  - Nucleus is no longer treated as a structureless particle
  - Main contribution from **two-photon exchange** δ_{TPE}
  - **Nuclear excited states** become necessary

➡ δ_{TPE} contributes at $(Z\alpha)^5$

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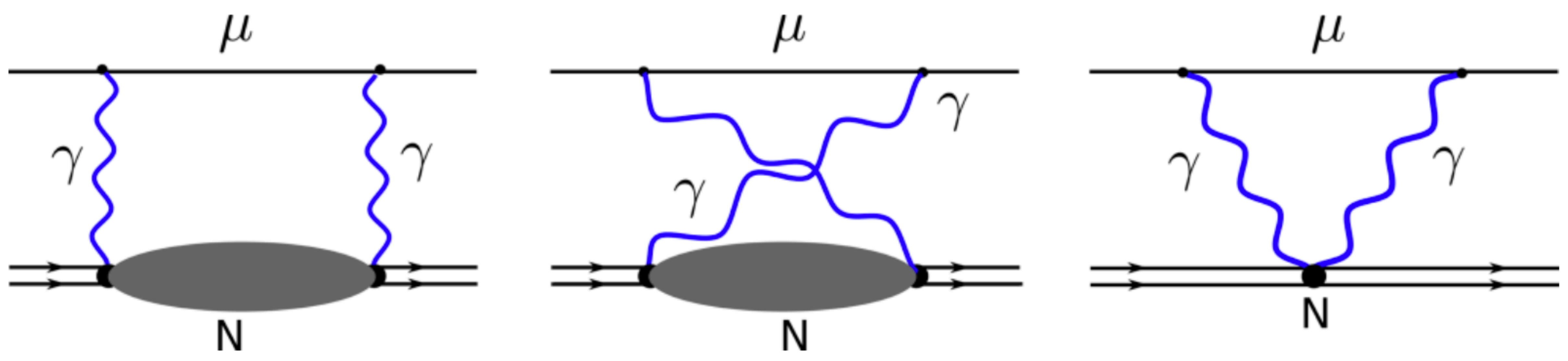
Two photon exchanges contributions



$$\Delta E_{nl} = -\frac{(4\pi Z\alpha)}{m_r}|\phi_{nl}(0)|^2 \mathrm{Im}\int \frac{d^4q}{(2\pi)^4} D^{\mu\rho}(q)D^{\nu\tau}(-q)\ t_{\mu\nu}(q,k)\ T_{\rho\tau}(q,-q)$$

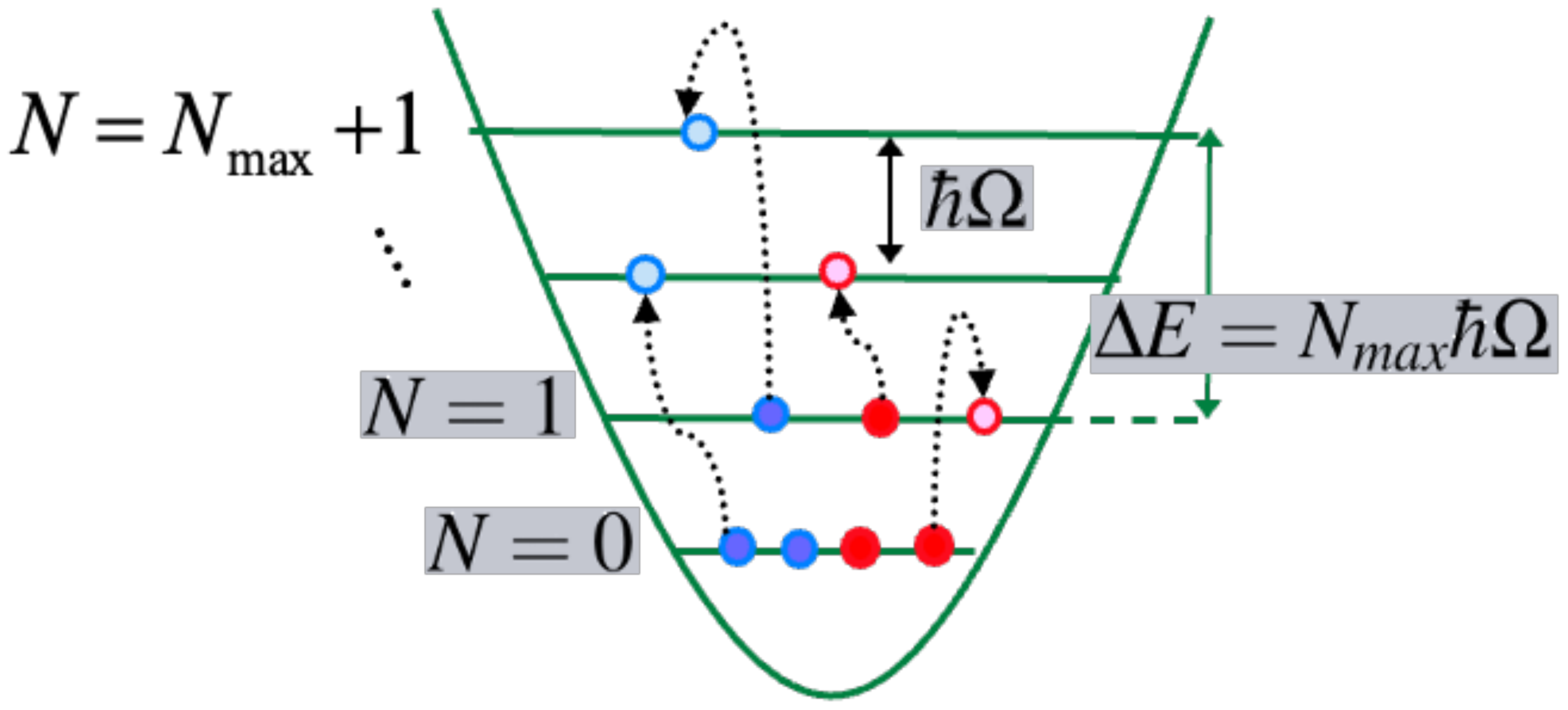
with:

- $D^{\mu\nu}(q) \equiv$ the photon propagator
- $t_{\mu\nu} \equiv$ the leptonic tensor
- $T_{\mu\nu} \equiv$ the hadronic tensor
- $k \equiv (m_r, 0)$

[Bernabeu et al, Nuclear Physics A (1974)]
[Rosenfelder Nuclear Physics A (1983)]
[Hernandez et al. Physical Review C (2019)]

Ab initio No-Core Shell Model calculations

Anti-symmetrized products of
many-body HO states



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Lanczos tridiagonalization algorithm [Lanczos (1950)]

- Initialization: normalized pivot $|\phi_1\rangle$
- Recursion: α_i, β_i and $|\phi_i\rangle$
  - $\beta_{i+1}|\phi_{i+1}\rangle = H|\phi_i\rangle - \alpha_i|\phi_i\rangle - \beta_i|\phi_{i-1}\rangle$
  - $\alpha_i = \langle\phi_i|H|\phi_i\rangle$ and β_{i+1} st $\langle\phi_{i+1}|\phi_{i+1}\rangle = 1$
- Output:
  - Lanczos basis and coefficients $\{|\phi_i\rangle, \alpha_i, \beta_i\}$ → **H in Lanczos basis**

$$\begin{pmatrix} \alpha_1 & \beta_2 & & & & \\ \beta_2 & \alpha_2 & \beta_3 & & & \\ & \beta_3 & \alpha_3 & \ddots & & \\ & & \ddots & \ddots & \beta_{k-1} & \\ & & & \beta_{k-1} & \alpha_{k-1} & \beta_k \\ & & & & \beta_k & \alpha_k \end{pmatrix}$$

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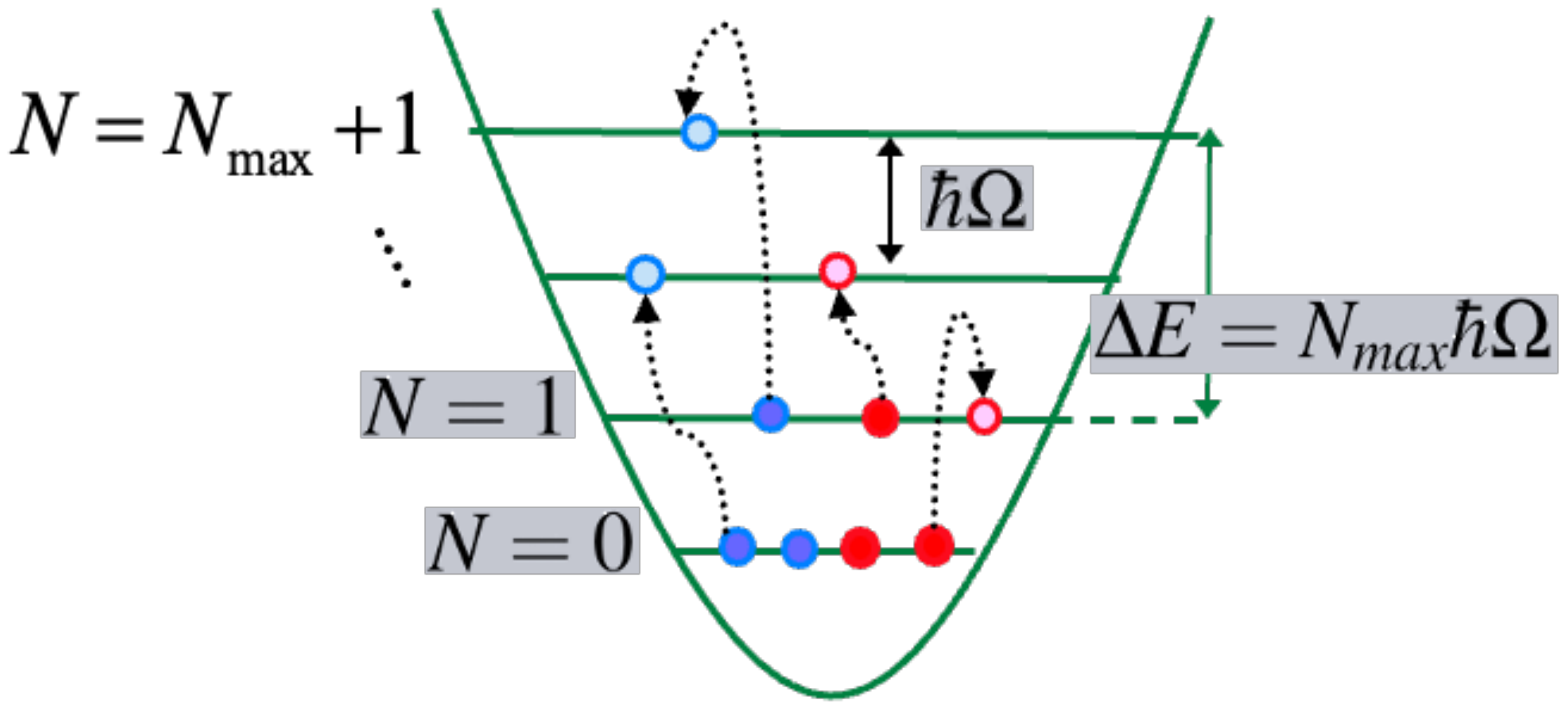

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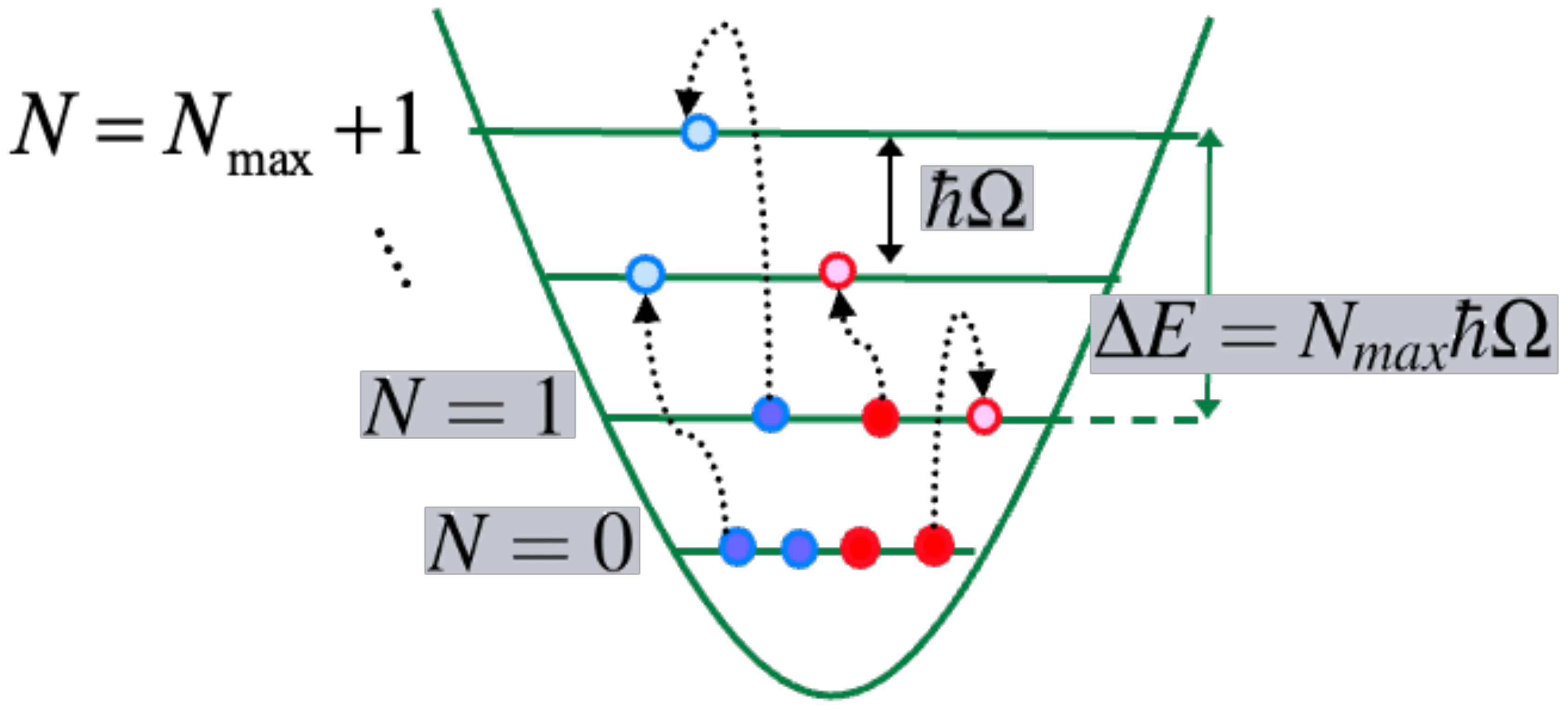

Application to nuclear structure

- Efficient calculation of spectra
  - Selection rules ⇒ **Fast matrix-vector multiplication**
  - In practice: $N_L \sim 100 - 200$ is sufficient
- Application to ^{7}Li
  - $N_L = 200$ for $N_{\max} = 1$ to 9
  - Ground-state of ^{7}Li $|\Psi\rangle$ ⇒ **Starting point for $\delta^A_{\rm pol}$**

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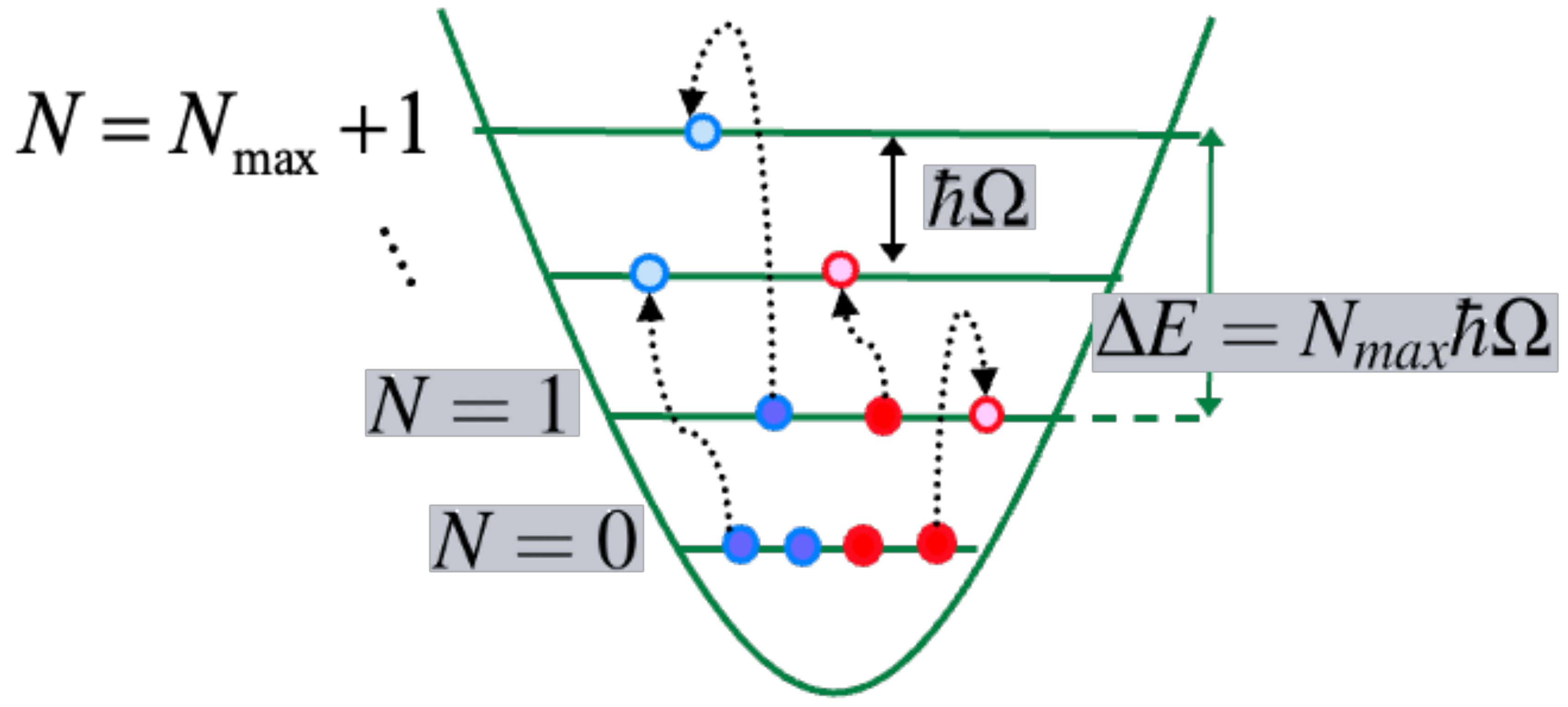

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Lanczos-strength algorithm for $\delta^A_{\rm pol}$

- Spectral function obtained with a second Lanczos:
  - Pivot based on $1^{\rm st}$ Lanczos output: $|\phi'_1\rangle = \dfrac{O|\Psi^{J^\pi T}\rangle}{\sqrt{\langle\Psi^{J^\pi T}|O^\dagger O|\Psi^{J^\pi T}\rangle}}$

  ➡ **Strengths:** $|\langle\Psi^{J^\pi T}_n|O|\Psi^{J^\pi T}\rangle|^2 = |\langle\phi'_1|\Psi^{J^\pi T}_n\rangle|^2\langle\Psi^{J^\pi T}|O^\dagger O|\Psi^{J^\pi T}\rangle$
- Convergence properties:
  - Recovers **exactly** the first $2n$ moments for n Lanczos steps!

  ➡ **One additional NCSM run per operator**

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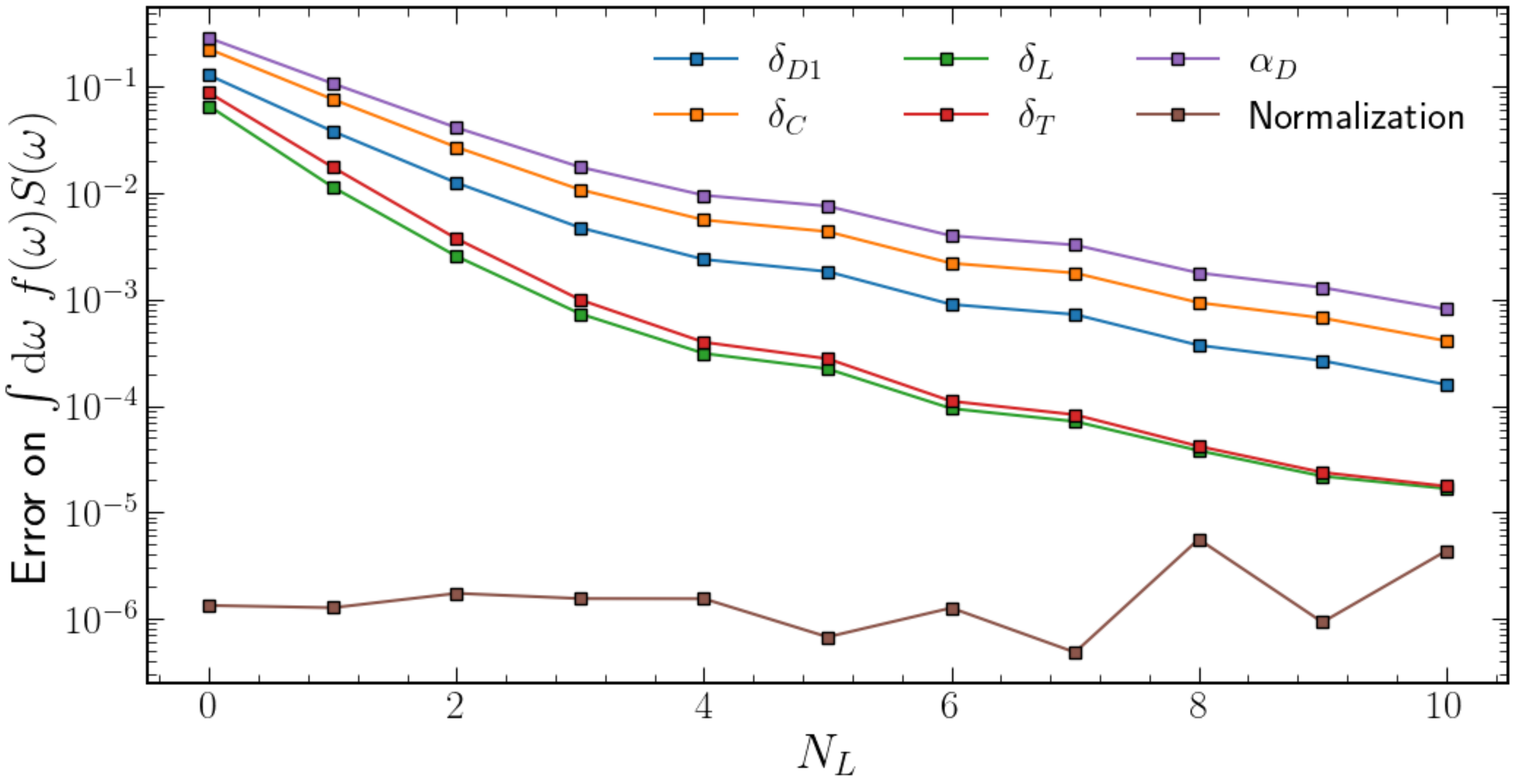

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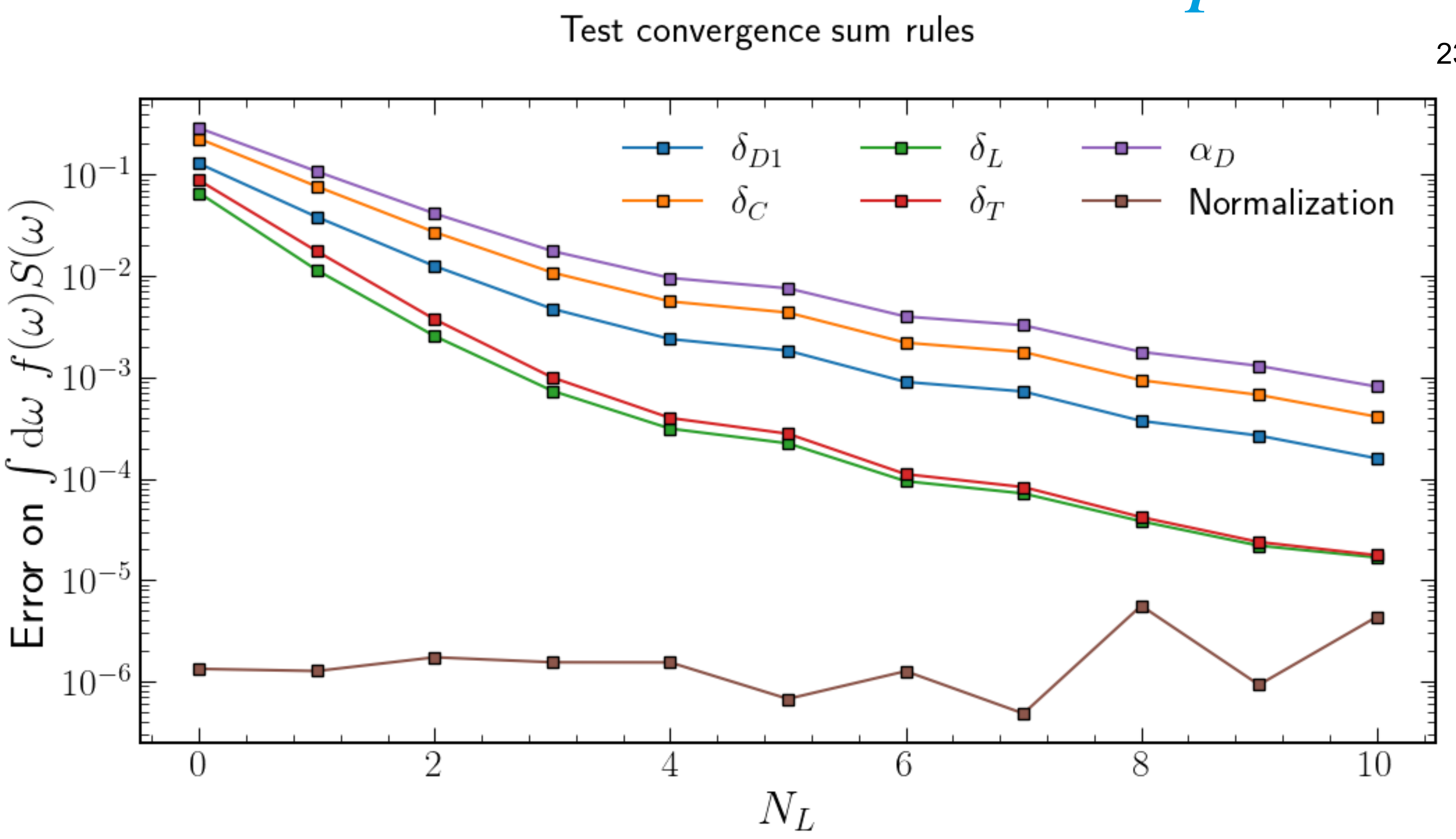

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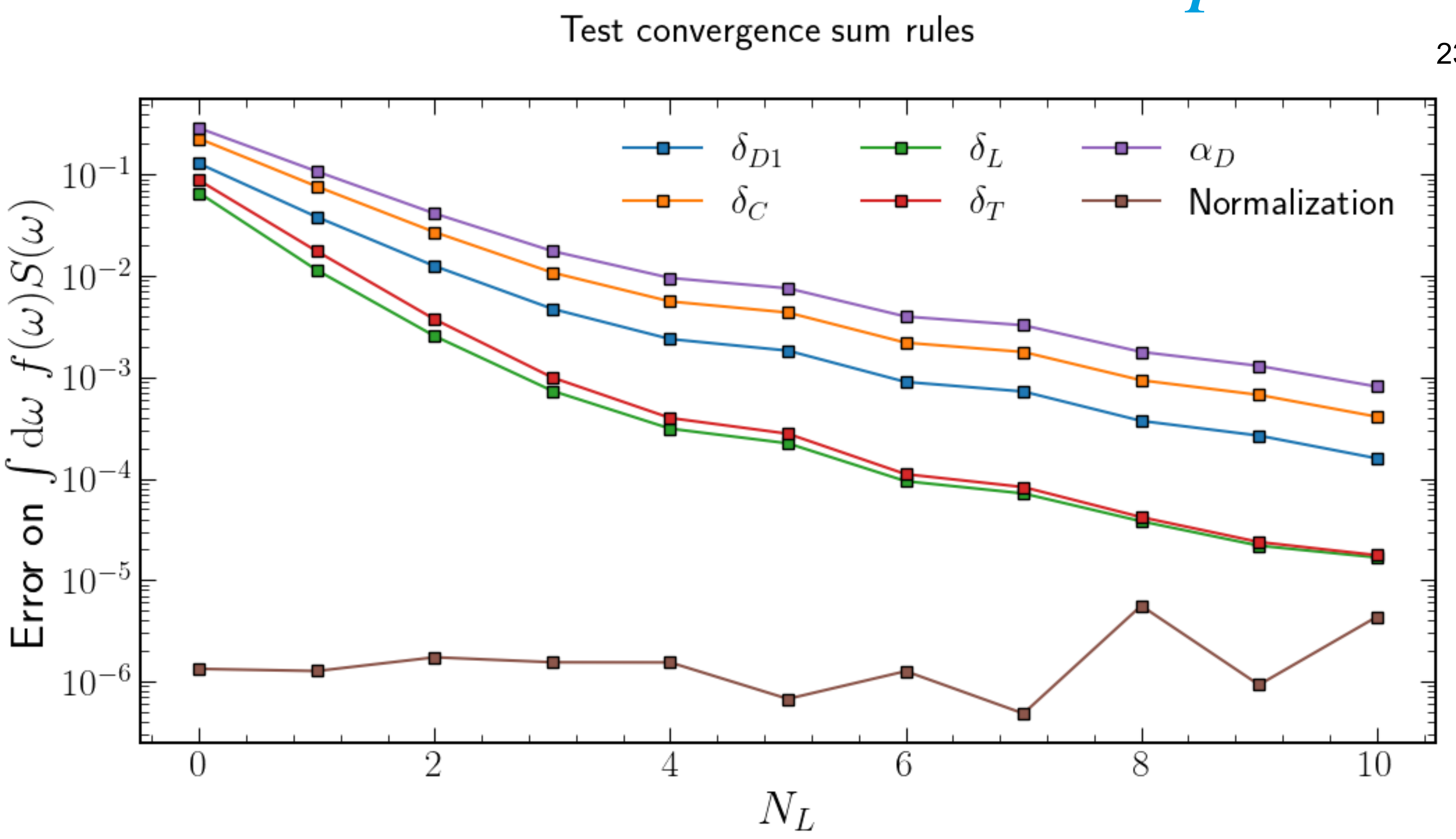

First conclusion: **numerical noise from Lanczos algo is negligible**

Next step: **q-dependent calculations of δ^A_{pol} !**