Using Functional Methods for EFT Calculations

Anders Eller Thomsen

Primarily based on [2412.12270] in collaboration with
J. Fuentes-Martín, *A. Moreno-Sánchez*, and *A. Palavrić*

SMEFT Meets ChEFT
TRIUMF, 29 September 2025

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Swiss National
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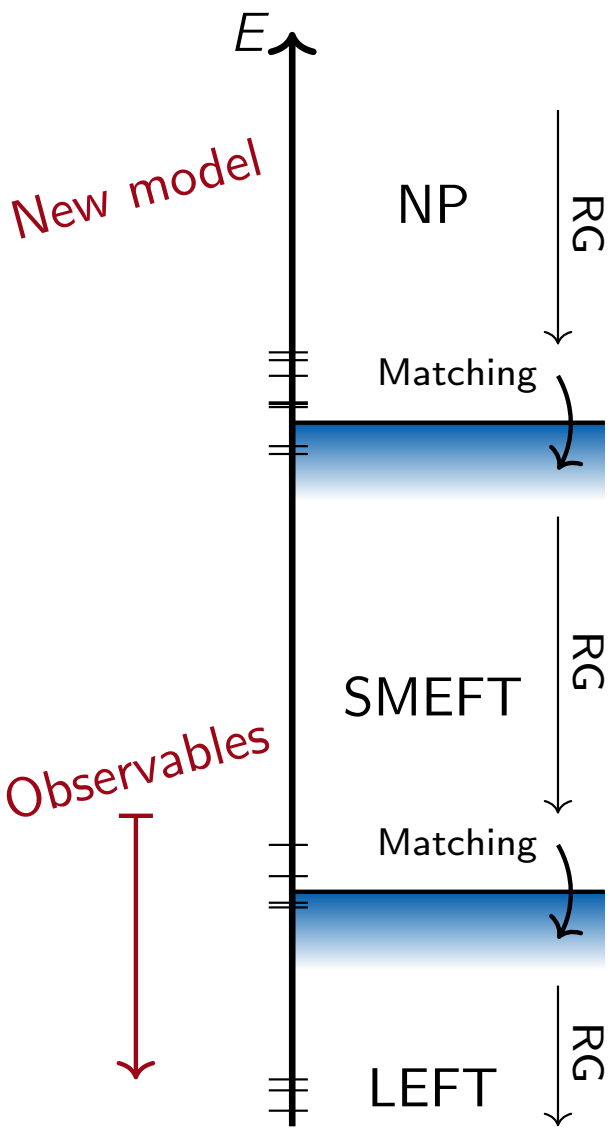
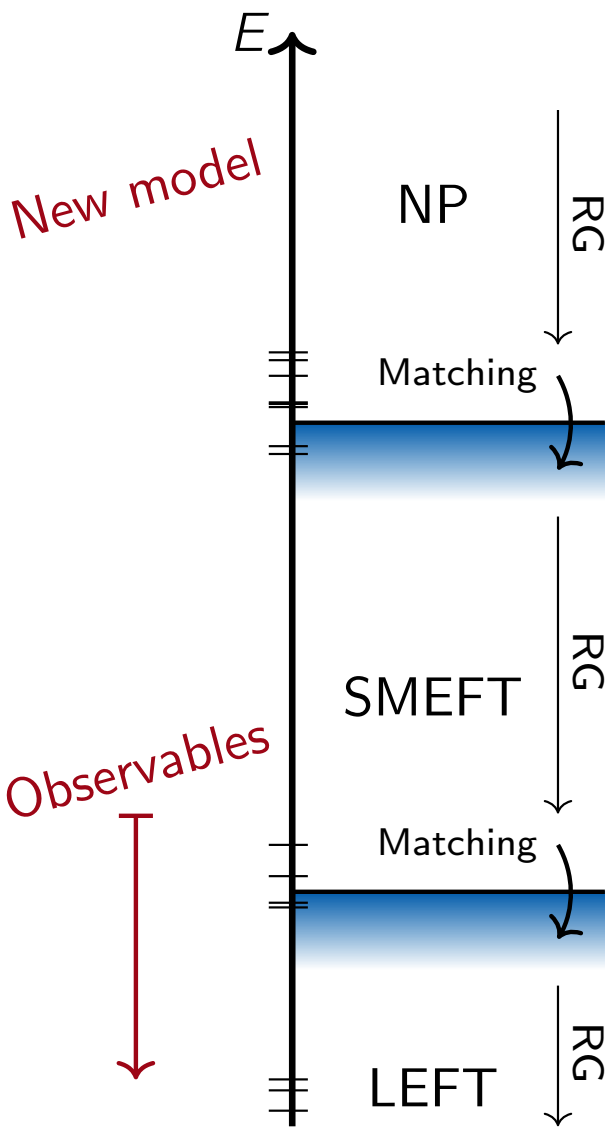
EFT workflow



Current status:

- Automated RG flow @ 1-loop
- Automated UV matching @ 1-loop
- Automated SMEFT likelihood
- Calculation of observables @ 1-loop order

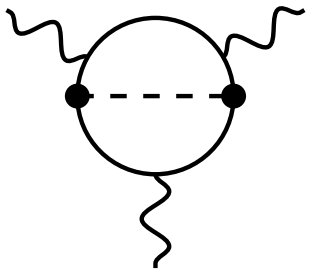
EFT workflow



Why we should go beyond one-loop order:

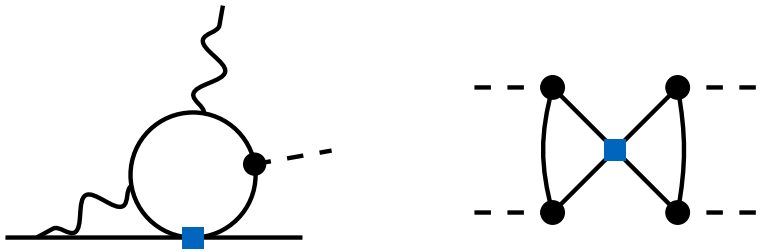
- Restoration of scheme invariance
- New effects at 2-loop order

$\mathcal{O}_{WW\widetilde{W}} \subseteq$



Das Bakshi *et al.* [2103.15861]

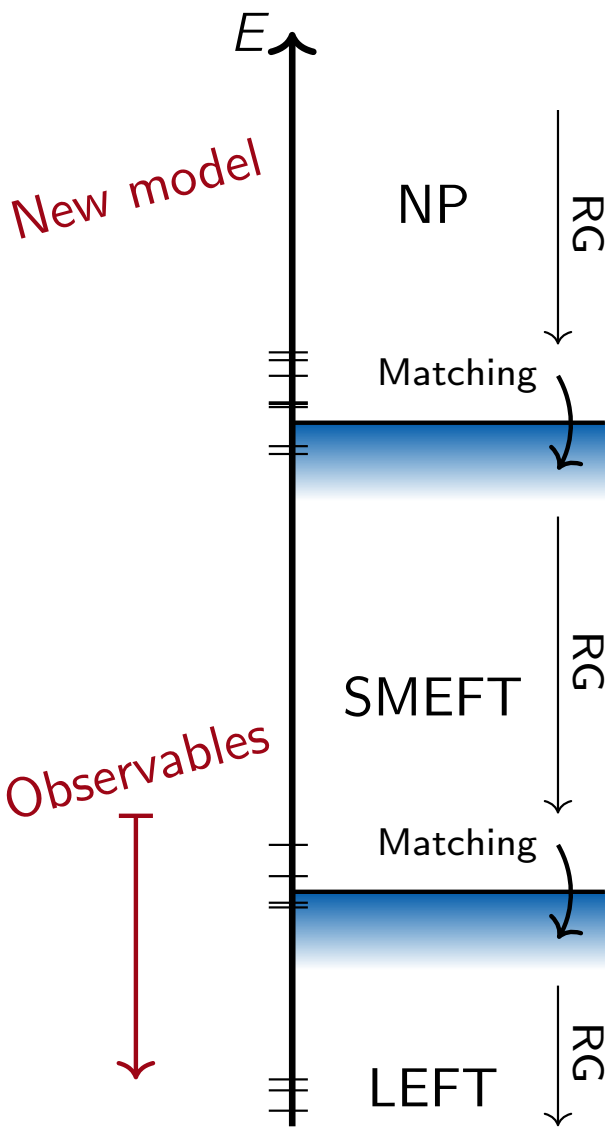
- Surprisingly large 2-loop effects



Ardu, Davidson [2103.07212]; Allwicher *et al.* [2302.11584]

- We might learn something new about QFT

EFT workflow



Functional calculations:

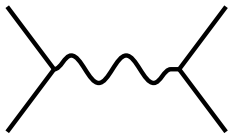
- Effective potential (momentum expansion)
- 1-loop **matching calculations** UOLEA, `Matchete`,...
- 2-loop RGs with Heat-Kernel
- **2-loop SMEFT RGs** w.i.p. (see talk by Javier Fuentes-Martín)

What are functional methods anyway?

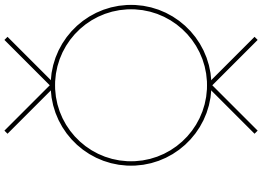
$$\mathcal{L}(\eta, D\eta)$$

Amplitudes

Functionals



$$\mathcal{W}[\mathcal{J}]$$

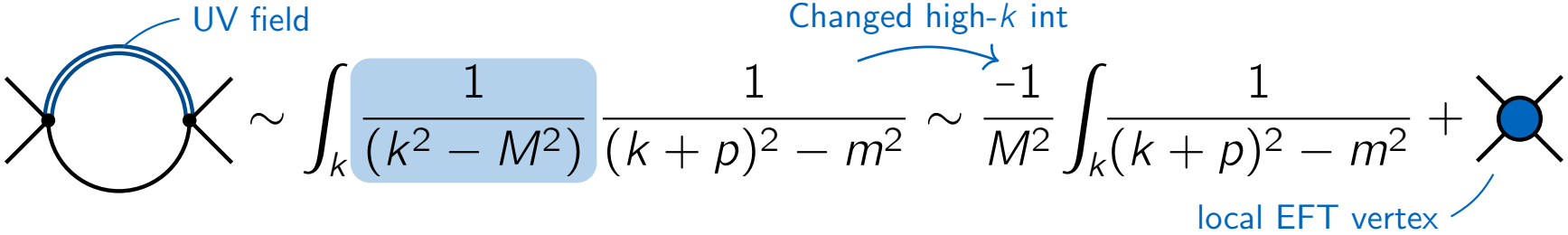


$$\Gamma[\hat{\eta}]$$

Quantum information

Where can we use functional methods?

- EFT matching—extraction of EFT couplings



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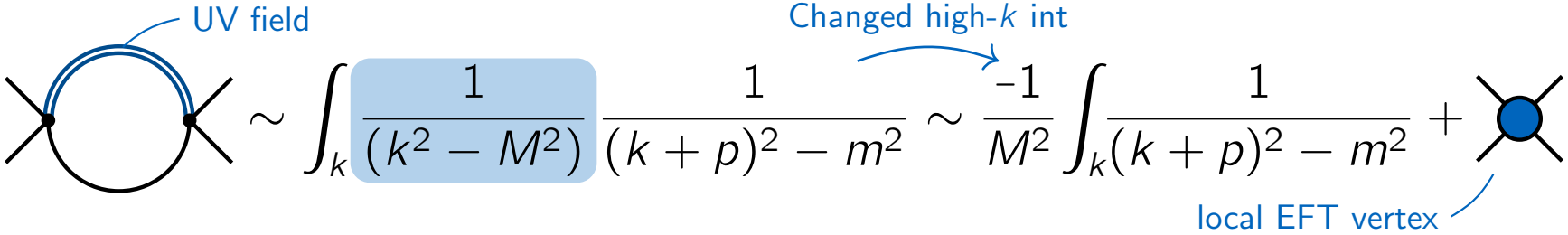
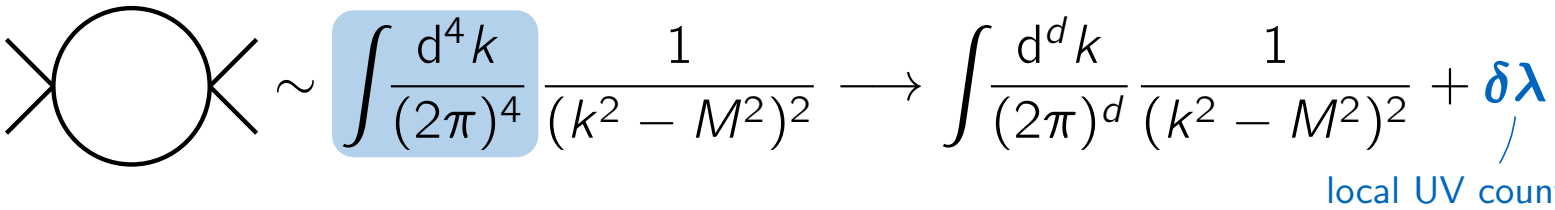


- Renormalization/RG functions—extraction of counterterms



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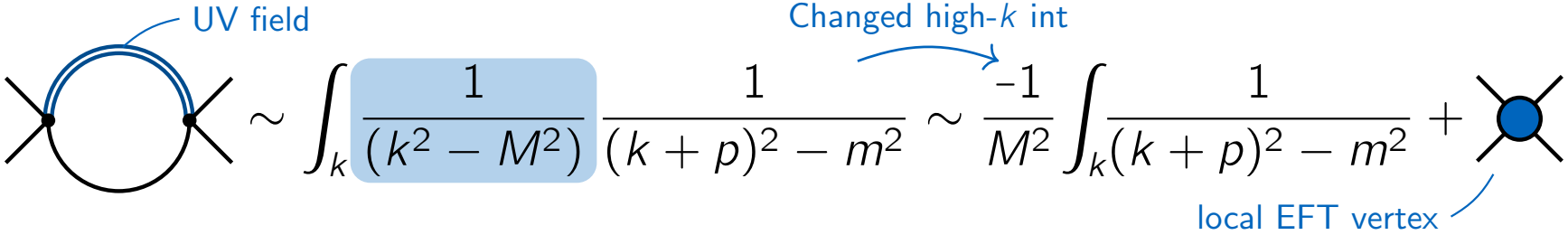

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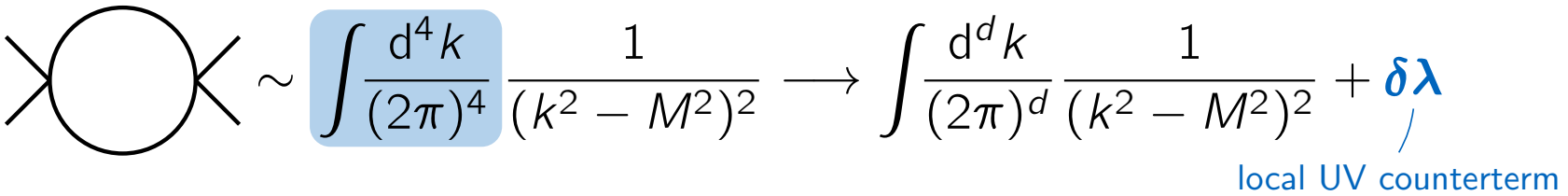

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  But there might be little point... and some *trouble* ⇜ non-local

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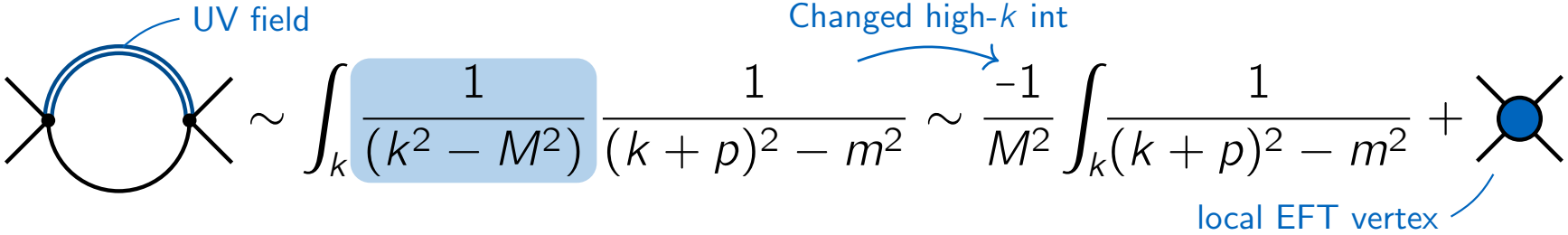

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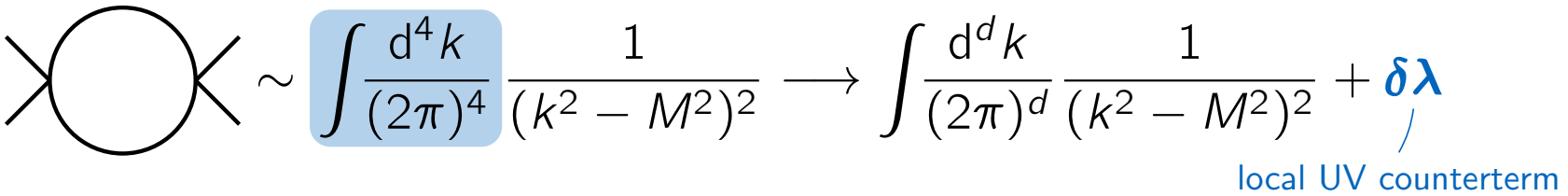

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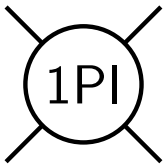
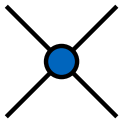
But what exactly are the advantages of functional methods?

A direct approach to RG quantities

Diagrammatic approach (tensor quantity):



$$= \int dx_1 e^{-ip_1\cdot x_1} \cdots \frac{\delta}{\delta\hat{\eta}(x_1)} \cdots \Gamma[\hat{\eta}]\Big|_{\hat{\eta}=0}$$



$$= \int dx_1 e^{-ip_1\cdot x_1} \cdots \frac{\delta}{\delta\hat{\eta}(x_1)} \cdots \delta S[\hat{\eta}]\Big|_{\hat{\eta}=0}$$

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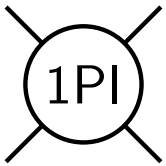
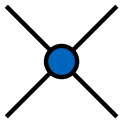


$$= \int \mathrm{d}x_1 e^{-ip_1\cdot x_1} \cdots \frac{\delta}{\delta\hat{\eta}(x_1)} \cdots \Gamma[\hat{\eta}]\Big|_{\hat{\eta}=0}$$



$$= \int \mathrm{d}x_1 e^{-ip_1\cdot x_1} \cdots \frac{\delta}{\delta\hat{\eta}(x_1)} \cdots \delta S[\hat{\eta}]\Big|_{\hat{\eta}=0}$$

Functional method (scalar quantity):

- EFT matching

$$S_{\mathrm{EFT}}[\hat{\phi}] \subseteq \Gamma_{\mathrm{UV}}[\hat{\eta}] \quad \longrightarrow \quad S_{\mathrm{EFT}} = \boldsymbol{T}_{\mathrm{hard}}\Gamma_{\mathrm{UV}}$$

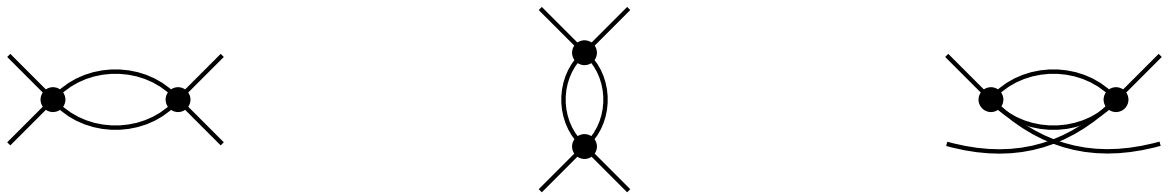
expansion operator

- Renormalization

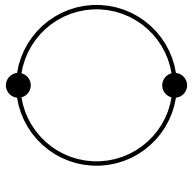
$$-\delta S[\hat{\eta}] \subseteq \Gamma[\hat{\eta}] \quad \longrightarrow \quad \delta S = -\boldsymbol{T}_{\delta}\Gamma$$

No repetition from symmetries

Diagrammatic approach: multiple permutations of external lines:

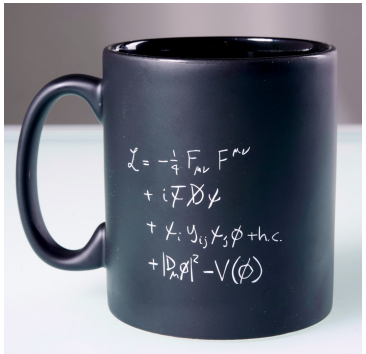


Functional method: contributions exclusively determined by internal lines

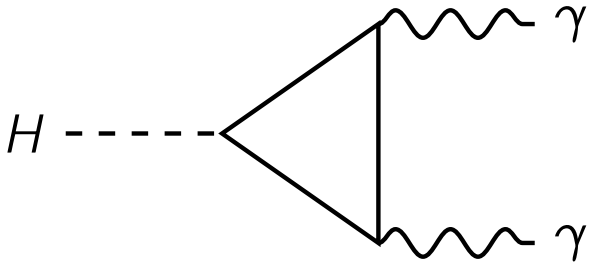


Saving more pronounced with more external lines or less internal symmetry, e.g., W^3, $(H^*H^*)^3$, $\phi^6, \ldots$

Benefits from a master formula



recipe $\longrightarrow$



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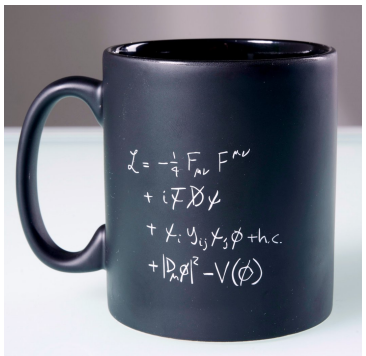

- **Bookkeeping:** organization by topology/internal propagators
- **Manifest covariance:** There is no need to separate out

$$D_\mu = \partial_\mu - iA_\mu$$

- **Recycling:** the methods recycle known loop-integral techniques

Functional calculation step-by-step

General formulas:

1. Expand quantum effective action (background field method)

2. Cast topologies as covariant momentum integrals

3. Determine expansion operator $\boldsymbol{T}_{\text{UV}}$ (what part of Γ to evaluate)

Computer implementation:

4. Expand the super diagram:
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5. Perform tadpole momentum integrals with known techniques

6. Simplify resultant Lagrangian

Semi-classical approximation

The **quantum effective action** is the generating functional of 1PI functions:

$$e^{i\Gamma_{\rm UV}[\hat{\eta}]} = \int \mathcal{D}\eta \, \exp\left(iS_{\rm UV}[\eta+\hat{\eta}] + iJ_I\eta^I \right), \qquad \frac{\delta\Gamma[\hat{\eta}]}{\delta\hat{\eta}^I} = -J_I$$

$J_I\eta^I \to \int_x J_a(x)\eta^a(x)$

The action is expanded by legs and loop order (including counterterms):

$$S[\eta+\hat{\eta}] = \widehat{S} + \sum_{\ell=0}^{\infty} \hbar^\ell \left(\eta_I \widehat{\mathcal{V}}_I^{(\ell)} + \tfrac{1}{2}\eta_I \widehat{\mathcal{V}}_{IJ}^{(\ell)} \eta_J + \tfrac{1}{6}\eta_I\eta_J\eta_K \widehat{\mathcal{V}}_{IJK}^{(\ell)} + \tfrac{1}{24}\eta_I\eta_J\eta_K\eta_L \widehat{\mathcal{V}}_{IJKL}^{(\ell)} + \dots \right)$$

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With a **saddlepoint approximation**

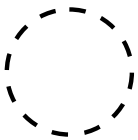
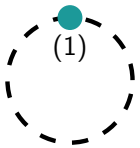
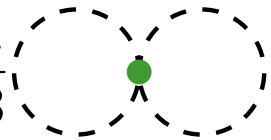
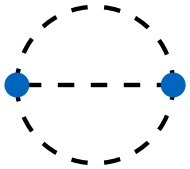
$$e^{i\Gamma_{\rm UV}[\hat\eta]} = e^{i\widehat{S}}\int\mathcal{D}\eta\,\exp\Big(\frac{i}{2}\eta^I\underbrace{\widehat{\mathcal{Q}}_{IJ}}_{\mathcal{V}_{IJ}^{(0)}}\eta^J\Big)\left[1+\frac{i}{24}\eta_I\eta_J\eta_K\eta_L\widehat{\mathcal{V}}_{IJKL}^{(0)} - \frac{1}{72}\big(\eta_I\eta_J\eta_K\widehat{\mathcal{V}}_{IJK}^{(0)}\big)^2+\dots\right]$$

Semi-classical approximation

The loop-expansion is a polynomium of tensor contractions

$$\Gamma[\hat{\eta}] = \widehat{S} + \frac{i\hbar}{2}\overbrace{\zeta_{II}\left(\log\widehat{\mathcal{Q}}\right)_{II}}^{\mathrm{STr}\log\mathcal{Q}} + \frac{i\hbar^2}{2}\zeta_{II}\widehat{\mathcal{Q}}^{-1}_{IJ}\widehat{\mathcal{V}}^{(1)}_{JI} - \frac{\hbar^2}{8}\widehat{\mathcal{Q}}^{-1}_{IJ}\widehat{\mathcal{Q}}^{-1}_{KL}\widehat{\mathcal{V}}^{(0)}_{IJKL}$$
$$+ \frac{\hbar^2}{12}\underbrace{\zeta_{IM}\zeta_{JN}\zeta_{IN}}_{\text{Spin statistics: } \pm 1}\widehat{\mathcal{Q}}^{-1}_{LI}\widehat{\mathcal{Q}}^{-1}_{MJ}\widehat{\mathcal{Q}}^{-1}_{NK}\widehat{\mathcal{V}}^{(0)}_{IJK}\widehat{\mathcal{V}}^{(0)}_{LMN} + \mathcal{O}(\hbar^3)$$

or as supergraphs

$$\Gamma_{\mathrm{UV}} = S_{\mathrm{UV}} + \frac{i}{2}\log$$  $$+ \frac{i}{2}$$  $$- \frac{1}{8}$$  $$+ \frac{1}{12}$$  $$+ \mathcal{O}(\hbar^3)$$

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or as supergraphs

$$\Gamma_{\mathrm{UV}} = S_{\mathrm{UV}} + \frac{i}{2}\log\ + \frac{i}{2}\ ^{(1)}\ - \frac{1}{8}\ + \frac{1}{12}\ + \mathcal{O}(\hbar^3)$$

All **vertices and propagators are fully dressed** with background fields

$$\mathcal{Q}_{IJ}[\hat{\eta}] = \frac{\delta^2 S^{(0)}}{\delta\eta^I\delta\eta^J}[\hat{\eta}] = \Big(\ \Big)^{-1}$$

$$\mathcal{V}^{(0)}_{IJK}[\hat{\eta}] =$$

The functional index

$$I = x, \quad \begin{cases} \text{scalar} : () & \begin{cases} H & : \quad (i) \\ H^* & : \quad (i) \end{cases} \\ \text{vector} : (\mu) & \begin{cases} B : & () \\ \vdots \end{cases} \\ \text{spinor} : (\alpha) & \begin{cases} q & : \quad (c,\ i,\ p) \\ q^C & : \quad (c,\ i,\ p) \\ \vdots \end{cases} \\ \text{(anti-)ghost} : () & \begin{cases} \vdots \end{cases} \end{cases}$$

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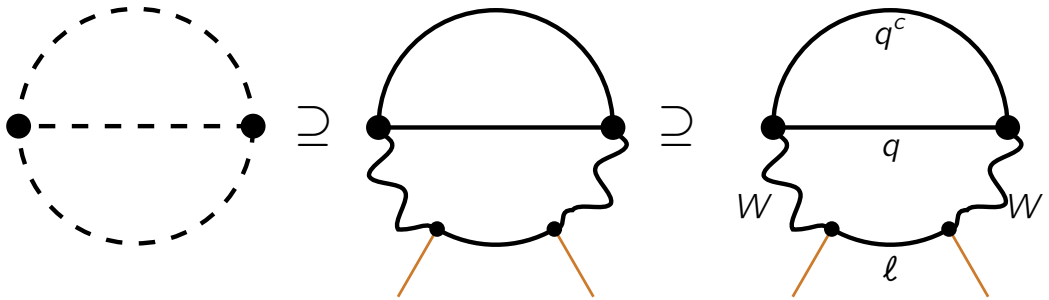
Individual diagrams are embedded in supergraphs

$$\widehat{\mathcal{Q}}^{-1}_{LI}\widehat{\mathcal{Q}}^{-1}_{MJ}\widehat{\mathcal{Q}}^{-1}_{NK}\widehat{\mathcal{V}}^{(0)}_{IJK}\widehat{\mathcal{V}}^{(0)}_{LMN} =$$



1:1 correspondence fails for external gauge bosons

Functional calculation step-by-step

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Covariant functional Feynman rules

Locality of the action: functional tensors factors as differential operators

$$\mathcal{V}_{I_1\ldots I_n} = \sum_{m_2\ldots m_n} V^{(\underline{m}_2,\ldots,\underline{m}_n)}_{a_1 a_2'\ldots a_n'}(x_1) \prod_{i=2}^{n} P_{x_i}^{\underline{m}_i} \delta_{a_i' a_i}(x_1, x_i)$$

$$\mathcal{Q}_{IJ} = \underbrace{Q_{ac}(x, \overbrace{P_x}^{iD_\mu})\delta^c{}_b(x,y)}_{\text{manifestly covariant}}$$

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$$\mathcal{Q}_{IJ} = Q_{ac}(x, \overbrace{P_x}^{iD_\mu})\delta^c{}_b(x,y)$$

And covariance is manifest with the **covariant delta functions**

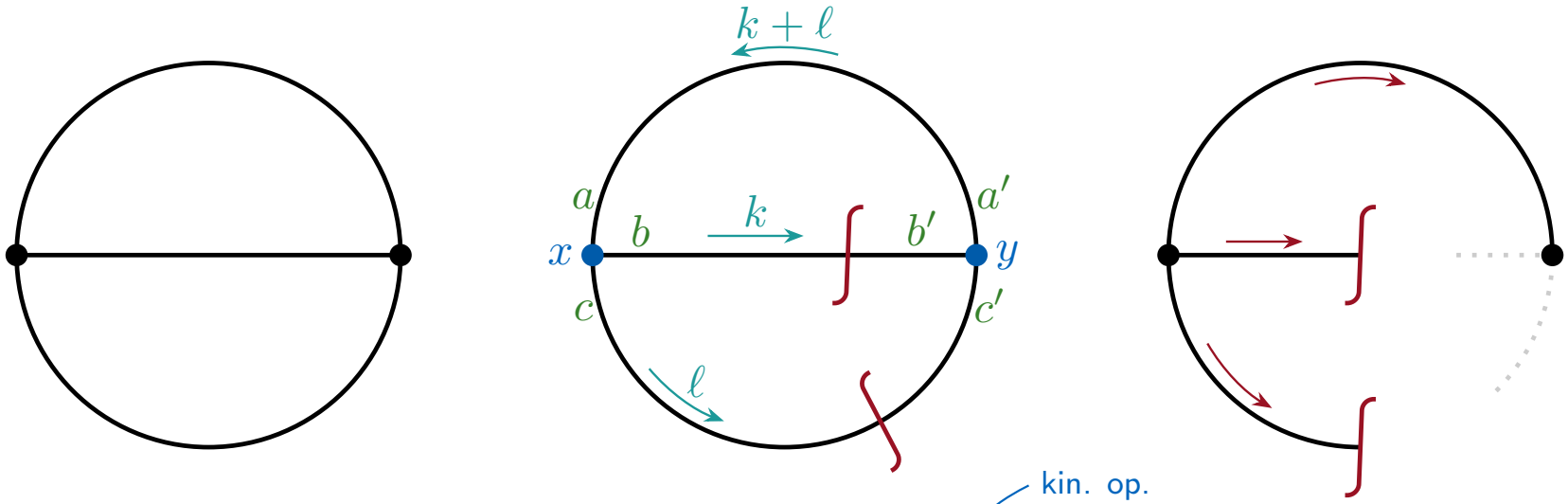
$$\frac{\delta\eta^a(x)}{\delta\eta^b(y)} = \delta^a{}_b(x,y), \qquad \delta^a{}_b(x,y) \equiv \overbrace{U^a{}_b(x,y)}^{\text{Wilson line}}\delta(x-y)$$

$$D^\mu_x \delta^a{}_b(x,y) = -D^\mu_y \delta^a{}_b(x,y)$$

The **background gauge field is a connection** in the field manifold

$$f^a(x) = U^a{}_b(x,y) f^b(y), \qquad U(x,y) = \mathscr{P}\exp\left[i\int_x^y \mathrm{d}z^\mu \widehat{A}_\mu(z)\right]$$

Sunset formula



$$E^{(\underline{m}_i,\underline{n}_j)}_{a_i b_j}(x, P_x + k) \equiv (-1)^{m_i} (P_x + k)^{\overleftarrow{m}_i} Q^{-1}_{a_i b_j}(x, P_x + k)(P_x + k)^{\underline{n}_j}$$

kin. op.

cov. devs. from vertices

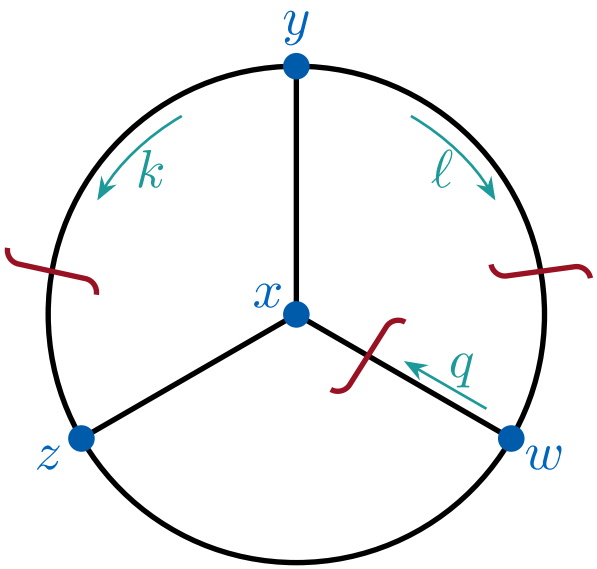
Sunset master formula:

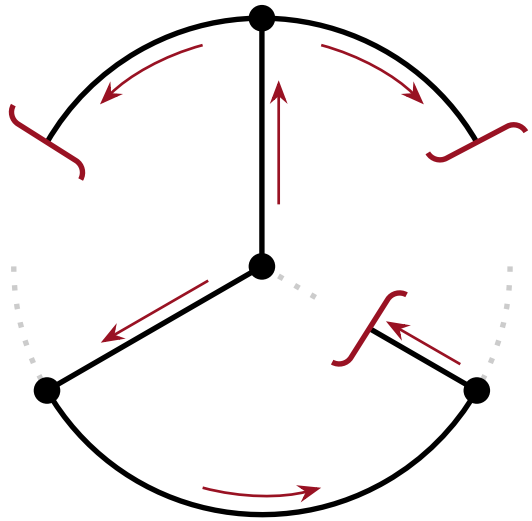
$$\frac{\hbar^2}{12} \sum_{m_i,n_i} \int_x \int_{k\ell} E^{(0,0)}_{aa'}(y, P_y - k - \ell) V^{(\underline{m}_1,\underline{n}_1)}_{abc}(x) V^{(\underline{m}_2,\underline{n}_2)}_{a'b'c'}(y)$$
$$\times [E^{(\underline{m}_1,\underline{m}_2)}(x, P_x + k) U(x,y)]_{bb'} [E^{(\underline{n}_1,\underline{n}_2)}(x, P_x + \ell) U(x,y)]_{cc'} \Big|_{y=x}$$

Grassmanian factor: $\zeta_{aa'}\zeta_{bb'}\zeta_{cc'}\zeta_{ab}\zeta_{ac}\zeta_{a'b}\zeta_{a'c}\zeta_{bc'}\zeta_{a'b'}\zeta_{a'c'} = \pm 1$

Tetrahedron formula

Systematic approach to deriving master formulas from diagram topologies:





$$\frac{i\hbar^3}{24}\sum_{m_i,n_i}\int_x\int_{k\ell q} E^{(0,0)}_{a_1 b_1}(y, P_y+k+\ell)\, E^{(\underline{m}_1,0)}_{a_2 c_1}(z, P_z+q-k-\ell)\, E^{(\underline{m}_3,\underline{n}_4)}_{c_2 d_3}(w, P_w+q-\ell)$$

$$\times\; V^{(\underline{m}_1,\underline{n}_1)}_{a_1 a_2 a_3}(x)\, V^{(\underline{m}_2,\underline{n}_2)}_{b_1 b_2 b_3}(y)\, V^{(\underline{m}_3,\underline{n}_3)}_{c_1 c_2 c_3}(z)\, V^{(\underline{m}_4,\underline{n}_4)}_{d_1 d_2 d_3}(w) \Big[E^{(\underline{m}_2,\underline{n}_3)}(y, P_y+k)\, U(y,z)\Big]_{b_2 c_3}$$

$$\times\; \Big[E^{(\underline{n}_2,\underline{m}_4)}(y, P_y+\ell)\, U(y,w)\Big]_{b_3 d_2} \Big[E^{(0,\underline{n}_1)}(w, P_w+q)\, U(w,x)\Big]_{d_1 a_3}\Big|_{y,z,w=x}$$

Functional calculation step-by-step

General formulas:

1. Expand quantum effective action (background field method)
2. Cast topologies as covariant momentum integrals
3. Determine expansion operator $\boldsymbol{T}_{\text{UV}}$ (what part of Γ to evaluate)

Computer implementation:

4. Expand the super diagram:
   - Expand kinetic operators by spin (scalar, fermion, vector)
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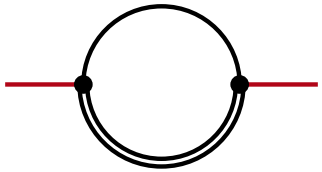
Expansion by regions

With expansion by regions we can **separate scales in loop integrals**, e.g., a 2-point function with $p^2, m^2 \ll \Lambda^2$:

$$I = \quad = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k+p)^2 - m^2} \frac{1}{k^2 - \Lambda^2}$$



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$k^2 \ll \Lambda^2$

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In dim. reg. integrals equal the sum of their **hard** and **soft** regions

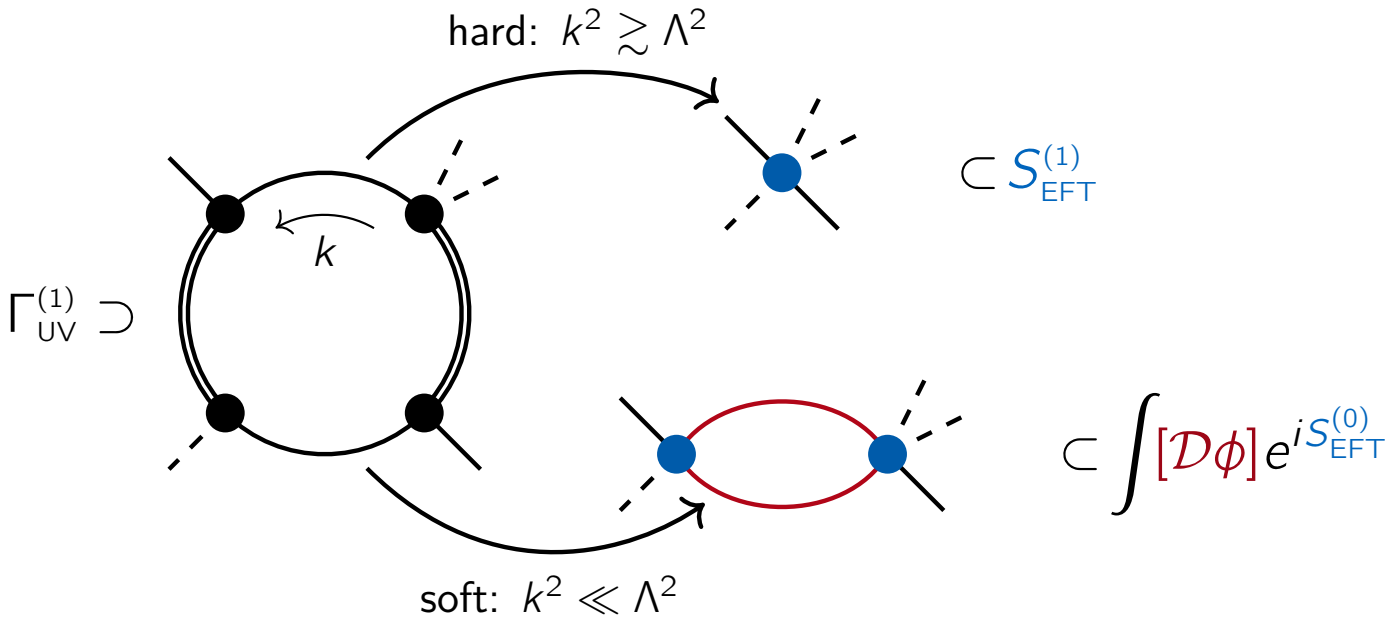
Beneke, Smirnov [hep-ph/9711391]; Jantzen [1111.2589]

$$I = I_h + I_s$$

The regions I_h and I_s are **systematically improvable** power series in $1/\Lambda^2$

Separation of scales in one-loop matching

Mixed (heavy–light) loop example:



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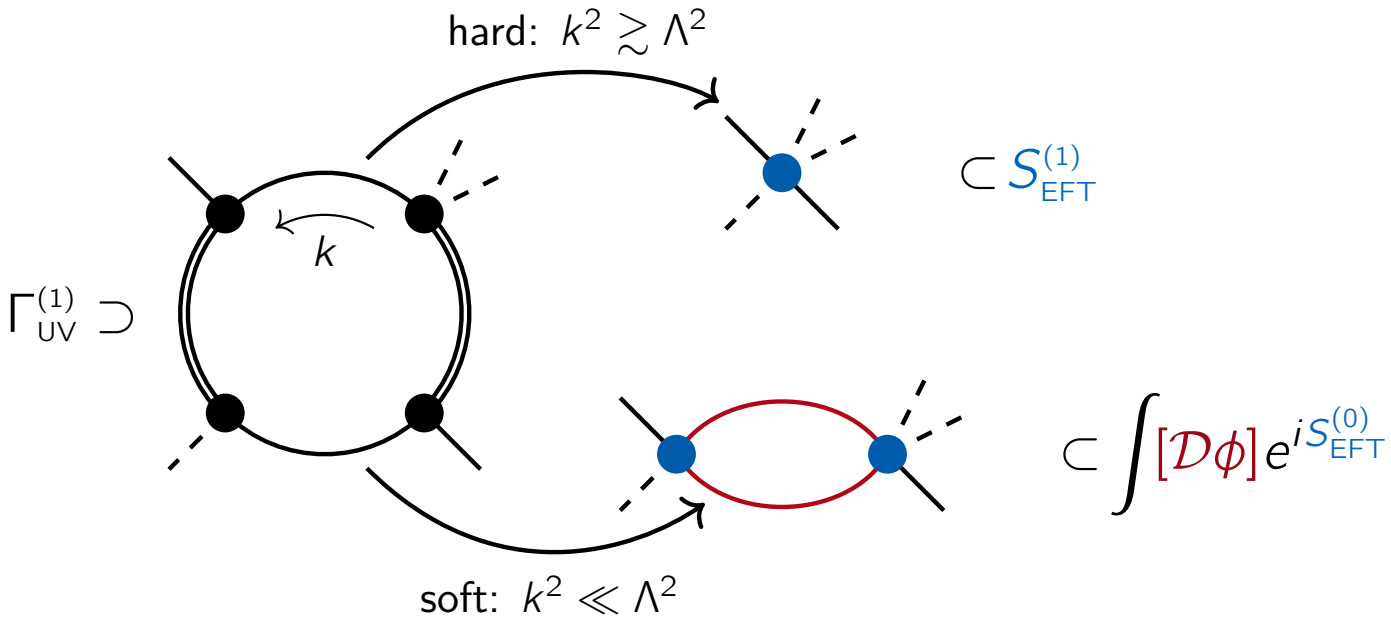

- $\Gamma^{(1)}_{\text{UV}}\big|_{\text{soft}}$: long-distance contributions included in 1-loop matrix elements of tree-level EFT operators

$$\Gamma^{(1)}_{\text{UV}}\big|_{\text{soft}} = \Gamma^{(1)}_{\text{EFT}}$$

- $\Gamma^{(1)}_{\text{UV}}\big|_{\text{hard}}$: short-distance contributions going into the EFT operators

Fuentes-Martin *et al.* [1607.02142]; Zhang [1610.00710]

$$\Gamma^{(1)}_{\text{UV}}\big|_{\text{hard}} = S^{(1)}_{\text{EFT}}$$

Matching formula

Hard-region matching formula

$$S_{\text{EFT}}[\phi] = \Gamma_{\text{UV}}[\widehat{\Phi}, \phi]\big|_{\text{hard}}, \qquad \frac{\delta \Gamma_{\text{UV}}|_{\text{hard}}}{\delta \Phi}[\widehat{\Phi}, \phi] = 0$$

"hard" denotes the part without *any* soft loop momenta (it includes all tree-level contributions) Fuentes-Martín, Moreno-Sánchez, Palavrić, **AET** [2311.13630] [2412.12270]

See also Fuentes-Martín *et al.* [1607.02142]; Zhang [1610.00710]

- Allows for matching without calculating any EFT loops
- The expansion operator $\boldsymbol{T}_{\text{UV}} = \boldsymbol{T}_{\text{hard}}$ selects a local part of Γ_{UV} to evaluate
- Proven to all orders in the loop expansion

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Evaluating momentum-space topologies

E.g., the sunset formula:

$$\frac{\hbar^2}{12}\sum_{m_i,n_i}\int_x\int_{k\ell} E^{(0,0)}_{aa'}(y,P_y-k-\ell)\,V^{(\underline{m}_1,\underline{n}_1)}_{abc}(x)V^{(\underline{m}_2,\underline{n}_2)}_{a'b'c'}(y) \times [E^{(\underline{m}_1,\underline{m}_2)}(x,P_x+k)\,U(x,y)]_{bb'}[E^{(\underline{n}_1,\underline{n}_2)}(x,P_x+\ell)\,U(x,y)]_{cc'}\Big|_{y=x}$$

RG calculations with $\boldsymbol{T}_{\text{UV}}$ (k,ℓ large) call for expansion of the propagators:

$$\boldsymbol{T}_{\text{UV}}E^{(\underline{m},\underline{n})} \supset \boldsymbol{T}_{\text{UV}}Q^{-1} = \boldsymbol{T}_{\text{UV}}\Delta^{-1}\sum_{n=0}^{\infty}\left(X\Delta^{-1}\right)^n$$

- Typical propagator: $\Delta^{-1} \sim \dfrac{1}{(k+P_x)^2-m^2} = \dfrac{1}{k^2} - \dfrac{2k\cdot P_x}{k^4} + \ldots$
- Typical X-term: $X_{\phi A_\mu} \supset 2g\phi P_x^\mu$
- Closed formula: $P_x^{\mu_1}\cdots P_x^{\mu_m}U(x,y)\Big|_{y=x} = p_m\big(P_x,\,G_{\mu\nu}\big)$ (polynomial)

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Summary: **tensor contract; expand; loop-integrate**

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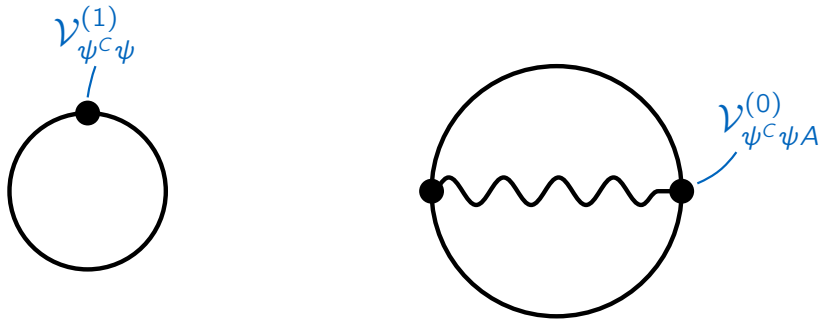
Example: Euler–Heisenberg

Let's integrate out the electron at 2-loop orders!



2-loop, EFT-order 8, by-hand calculation

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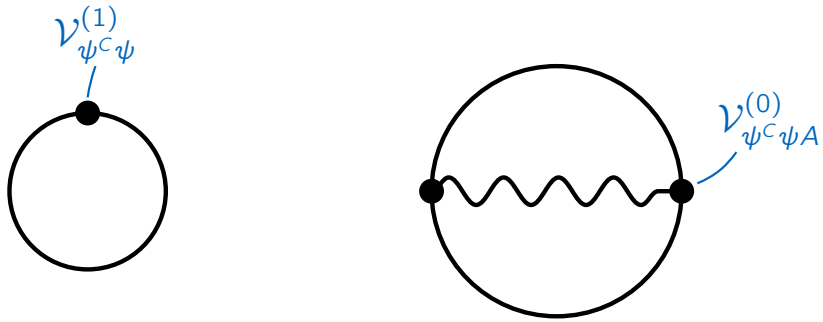

2-loop, EFT-order 8, by-hand calculation

agrees with Grozin [1212.5144]

$$\mathcal{L}_{\text{EFT}} = -\frac{1}{4e^2}\left[1+\frac{e^2}{16\pi^2}\frac{4}{3}\log\frac{\mu^2}{m^2}+\frac{e^4}{(16\pi^2)^2}\left(\frac{13}{3}-4\log\frac{\mu^2}{m^2}\right)\right]F^{\mu\nu}F_{\mu\nu}$$
$$+\left[\frac{1}{16\pi^2}\frac{7}{90}\frac{1}{m^4}-\frac{1}{(16\pi^2)^2}\frac{e^2}{m^4}\left(\frac{14}{15}\log\frac{\mu^2}{m^2}+\frac{247}{810}\right)\right](F^{\mu\nu}F_{\nu\sigma})^2$$
$$-\left[\frac{1}{16\pi^2}\frac{1}{36}\frac{1}{m^4}-\frac{1}{(16\pi^2)^2}\frac{e^2}{m^4}\left(\frac{1}{3}\log\frac{\mu^2}{m^2}-\frac{49}{324}\right)\right](F^{\mu\nu}F_{\mu\nu})^2$$

consistent with μ-independence

Fuentes-Martín, Moreno-Sánchez, Palavrić, **AET** [2412.12270]

Summary

- Functional methods are an excellent match for **all RG calculations**
- The parallels with ordinary Feynman diagrams ensure that we may borrow known integrals techniques
- The techniques are extendable to 3-loop orders and beyond (subject to the usual computational limitations)
- Other interesting results: all-order EFT matching formula
- Practical implementation of the techniques see talk by Javier Fuentes-Martín

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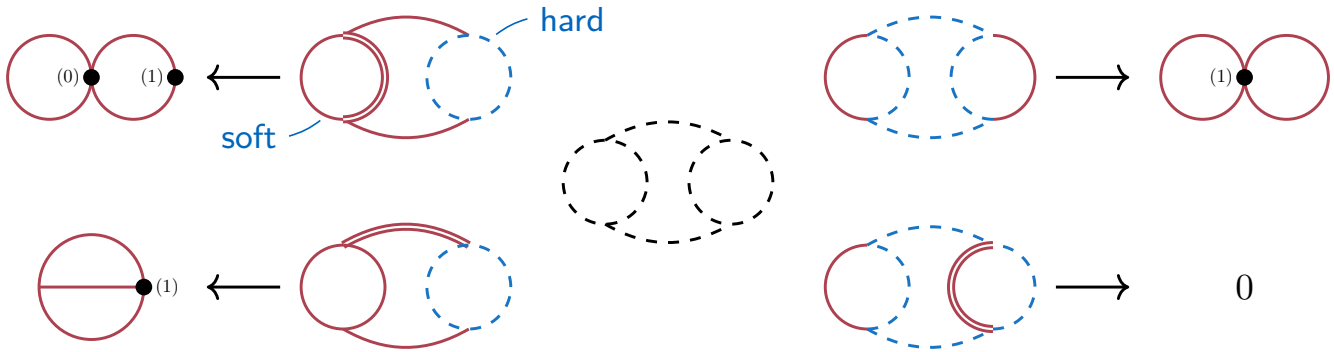
Thank you!

Backup

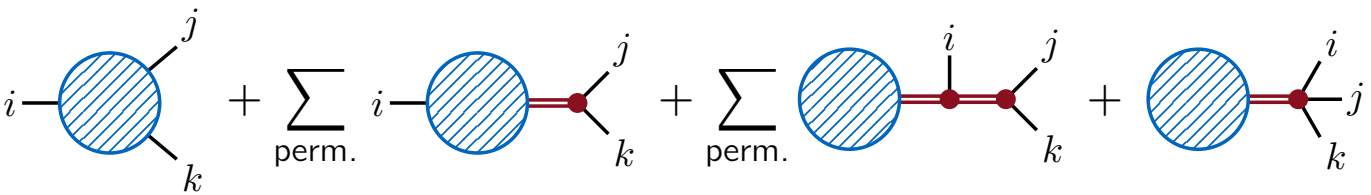
Proof of hard-region matching formula

- Expansion by regions (soft/hard) extended to arbitrary loop order
- Compare UV and EFT effective action (**induction in loop order**)
  - Decompose UV topologies (figure is not comprehensive)



  - Use EFT vertices produced from lower order matching

$$i\,\widehat{\mathcal{W}}^{(\ell)}_{ijk} =$$



- Account for combinatorics and symmetry factors

See Fuentes-Martín, Moreno-Sánchez, Palavrić, **AET** [2412.12270]