TRIUMF

Some subleading three nucleon interactions at light nuclei

Georgios Palkanoglou
TRIUMF

working with Petr Navratil

PAINT26

February 24, 2026

Discovery, accelerated

Outline

- Introduction
  - "Why any/more many-nucleon forces?"
- Leading ($\mathrm{N^2LO}$) and subleading ($\mathrm{N^4LO}$) 3-nucleon contacts
  - The 13 new contacts
  - The $T = 3/2$ projection
  - Effects on ^{3}H
- Quark-mass dependent 3NFs at light nuclei
  - Scaling of LECs and enhancement
  - Effects on ^{3}H and ^{4}He
  - Tuning strategy
- *Summary & Outlook*

Discovery, accelerated

History

All realistic *NN* forces underbind the triton [7], and small differences among them can be traced to nonlocalities.

[*Friar et al, PRC (1999)*]

Overbound nuclear matter (wrong saturation density)

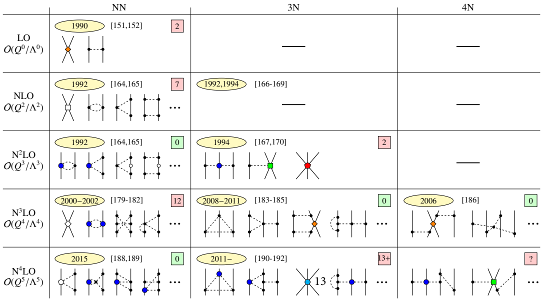
[*Akmal et al, PRC (1998)*] ++

- *Fujita-Miyazawa*: One of the earliest (1957)
- *UIX*: starting from Fujita-Miyazawa
- *Tucson-Melbourne* / *Brazil*

⋮

- Chiral 3NFs: $\mathrm{N^2LO}$ (1994), $\mathrm{N^3LO}$ ($\sim$ 2010), $\mathrm{N^4LO}$ (2011).

N4LO three-nucleon contacts



Adapted from [K. Hebeler, Phys. Rept. (2021)]

3NF LECs

$$N^2LO \;\; \#2 \;:\;\; c_D, c_E$$
$$N^3LO \;\; \#0$$
$$N^4LO \;\; \#13+ \;:\;\; E_i$$

[*Girlanda et al, PRC (2011)*]

[*Epelbaum et al, 2602.12879 (2026)*]

Discovery, accelerated

Why better 3NF?

Outstanding problems

- $\mathrm{A_y}$ ***puzzle***: long-standing underestimation of the polarization asymmetry in N-d scattering by $30\ \%$.

  Known since 1990s: lack of spin-orbit 3NFs.

- ***Space-star anomaly***: similar discrepancy in the break-up channel of N-d.

2NFs + modifications offer no resolution

3NFs up to $\mathrm{N^3LO}$ offer no resolution

[*Golak et al, EPJA (2014)*]

[*Witala et al, PRC (2021)*]

[*Girlanda et al, PRC (2019)*] +

Prospects

- **Scattering exp.**: d-p, p-^{3}He, etc. scattering can provide high quality data.

  [*K. Sekiguchi, FBS (2024)*]

  Fit 15 LECs for 3NF like the 21 LECs for 2NF (at $\mathrm{N^4LO}$).

  This will require controlled regulator effects, etc.

Outline

- Leading $(\mathrm{N}^2\mathrm{LO})$ and subleading $(\mathrm{N}^4\mathrm{LO})$ 3-nucleon contacts
    - The 13 new contacts
    - The $T = 3/2$ projection
    - Effects on ^{3}H

Leading ($\mathrm{N^2LO}$) three-nucleon contacts

There are 2 contacts: E (pure contact) and D (1-π exch. +contact.)

$$V_E = -\sum_{jk} E\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$$

$$V_D = -\sum_{ijk} \frac{g_A}{8f_\pi^2} D \frac{\boldsymbol{\sigma_j} \cdot \mathbf{q}_j}{\mathbf{q}_j^2 + M_\pi^2} (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)(\boldsymbol{\sigma}_i \cdot \mathbf{q}_j)$$

Tuned to binding energy ^{3}H and ^{4}He, or properties of light nucl., or GT ME of ^{3}H, or a_{nd} etc.

[*Nogga, et al, PRC (2006)*] [*Navràtil, et al, PRL (2007)*] [*Gazit, et al, PRL (2009)*] [*Epelbaum, et al, PRC (2002)*]

There are also c_1,c_3, and c_4 (2-π exch.)

$$V_{1,3,4} = \sum_{ijk} \frac{1}{2} \left(\frac{g_A}{2f_\pi}\right) \frac{(\boldsymbol{\sigma}_i \cdot \mathbf{q}_i)(\boldsymbol{\sigma}_j \cdot \mathbf{q}_j)}{(\mathbf{q}_i^2 + M_\pi^2)(\mathbf{q}_j^2 + M_\pi^2)} F_{ijk}^{\alpha\beta} \tau_i^\alpha \tau_j^\beta$$

where

$$F_{ijk}^{\alpha\beta} = \delta^{\alpha\beta} \left[-\frac{4c_1 M_\pi^2}{f_\pi^2} + \frac{2c_3}{f_\pi^2} \mathbf{q}_i \cdot \mathbf{q}_j\right] + \sum_\gamma \frac{c_4}{f_\pi^2} \epsilon^{\alpha\beta\gamma} \tau_k^\gamma \boldsymbol{\sigma}_k \left[\mathbf{q}_i \times \mathbf{q}_j\right]$$

(c_1,c_3, and c_4 enter the subleading 2-π exch. in the 2-nucleon potential.)

Subleading ($\mathrm{N}^4\mathrm{LO}$) three-nucleon contacts

Thirteen new contacts at $\mathrm{N}^4\mathrm{LO}$: ($\mathbf{q}_i = \mathbf{p}_i - \mathbf{p}_i'$):

$$O_1 = \sum_{i\neq j\neq k} \left(-q_i^2\right) , \quad O_2 = \sum_{i\neq j\neq k} \left(-q_i^2\right) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j , \quad O_3 = \sum_{i\neq j\neq k} \left(-q_i^2\right) \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j ,$$

$$O_4 = \sum_{i\neq j\neq k} \left(-q_i^2\right) \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$$

$$O_5 = \sum_{i\neq j\neq k} \left(-3\mathbf{q}_i \cdot \boldsymbol{\sigma}_i \mathbf{q}_i \cdot \boldsymbol{\sigma}_j + q_i^2 \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right) , \quad O_6 = \sum_{i\neq j\neq k} \left(-3\mathbf{q}_i \cdot \boldsymbol{\sigma}_i \mathbf{q}_i \cdot \boldsymbol{\sigma}_j + k_i^2 \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$$

$$O_7 = \sum_{i\neq j\neq k} \left(-\frac{i}{4}\mathbf{q}_i\right) \times (\mathbf{p}_i - \mathbf{p}_j) \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j) , \quad O_8 = \sum_{i\neq j\neq k} \left(-\frac{i}{4}\mathbf{q}_i\right) \times (\mathbf{p}_i - \mathbf{p}_j) \cdot (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$$

$$O_9 = \sum_{i\neq j\neq k} (-\mathbf{q}_i \cdot \boldsymbol{\sigma}_i \mathbf{q}_j \cdot \boldsymbol{\sigma}_j) , \quad O_{10} = \sum_{i\neq j\neq k} (-\mathbf{q}_i \cdot \boldsymbol{\sigma}_i \mathbf{q}_j \cdot \boldsymbol{\sigma}_j) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$$

$$O_{11} = \sum_{i\neq j\neq k} (-\mathbf{q}_i \cdot \boldsymbol{\sigma}_j \mathbf{q}_j \cdot \boldsymbol{\sigma}_i) , \quad O_{12} = \sum_{i\neq j\neq k} (-\mathbf{q}_i \cdot \boldsymbol{\sigma}_j \mathbf{q}_j \cdot \boldsymbol{\sigma}_i) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$$

$$O_{13} = \sum_{i\neq j\neq k} (-\mathbf{q}_i \cdot \boldsymbol{\sigma}_j \mathbf{q}_j \cdot \boldsymbol{\sigma}_i) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k .$$

[*Girlanda et al, PRC (2011)*]
[*Heusmann, 2602.12879 (2026)*] for longer range

$\mathrm{N}^4\mathrm{LO}$ three-nucleon contacts: Regulators

With a regulator

$$O \quad \rightarrow \quad F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; \Lambda) O F(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3; \Lambda)$$

Choices

$$\text{non-local regulator}: \quad F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; \Lambda) = \exp\left[-\frac{1}{4}\left(\pi_1^2 + \pi_2^2\right)^2 / \Lambda^4\right]$$

$$\text{local regulator}: \quad F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; \Lambda) = \prod_i \exp\left(-k_i^4 / \Lambda^4\right)$$

⋮

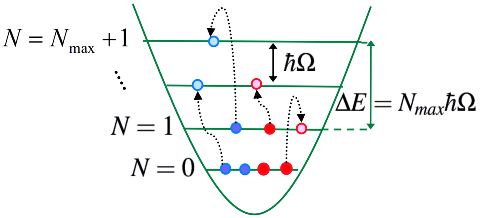
No Core Shell Model

Solution via a systematic expansion on A-body harmonic oscillator states.

Characterized by the HO frequency ($\hbar\Omega$, energy spacing of excitations) and number of excitations ($N_{\max}$).

[*Navratil et al, Phys.Rev.C* ***61*** *(2000)*]

[*Barrett et al. PPNP* ***69*** *(2013)*]



$$\left|\Psi_A^{J^\pi T}\right\rangle = \sum_{N=0}^{N_{\max}} \sum_n c_{Nn}^{J^\pi T} \left|\Phi_{Nn}^{J^\pi T}\right\rangle$$

PW decomposition

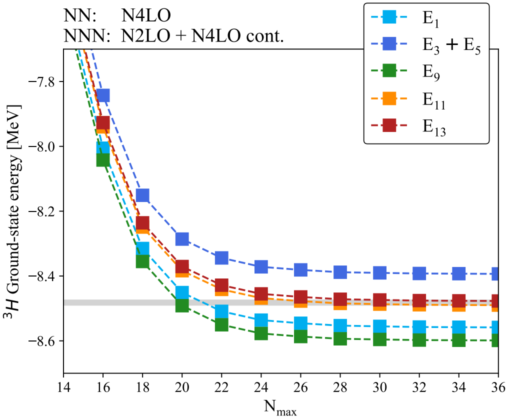
The Jj-coupled three-body partial wave basis
[*W. Glöckle (1983)*]+

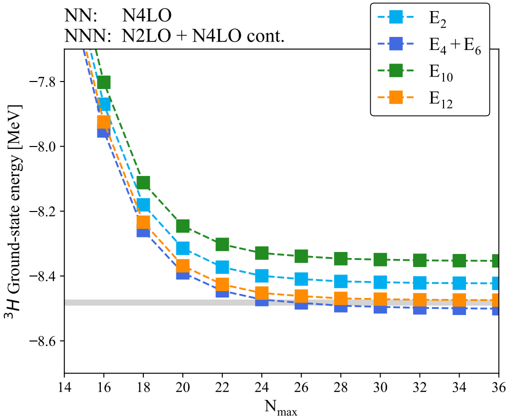
$$|N\alpha JT\rangle = |N[(ls)j(\mathcal{L}\mathcal{S})\mathcal{J}]JT(t\tau)\rangle$$

Channels specified by $J, T, \mathcal{P} = (-1)^{l+\mathcal{L}}$ and $\alpha = (l, s, j\mathcal{L}, \mathcal{J}, t)$.

$$\begin{aligned}\langle O_2\rangle = \delta_{ss'}&\left[E_1\delta_{tt'} + E_2\hat{t}\hat{t}'(-1)^{t+t'+T+\frac{1}{2}}\left\{\begin{array}{ccc} t & t' & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{array}\right\}\left\{\begin{array}{ccc} t & t' & 1 \\ \frac{1}{2} & \frac{1}{2} & T\end{array}\right\}\right]\times \\ &\times \hat{j}\hat{j}'\hat{\mathcal{J}}\hat{\mathcal{J}}'\hat{l}'\hat{\mathcal{L}}'(-1)^{J-\frac{1}{2}\mathcal{J}'-\mathcal{J}+l+\mathcal{L}+s}\times \\ &\times \sum_X (-1)^X \hat{X}^2 \left\{\begin{array}{ccc} l & s & j \\ j' & X & l'\end{array}\right\}\left\{\begin{array}{ccc} X & j & j' \\ J & \mathcal{J}' & \mathcal{J}\end{array}\right\}\left\{\begin{array}{ccc} \mathcal{L}' & 1/2 & \mathcal{J}' \\ \mathcal{J} & X & \mathcal{L}\end{array}\right\} C^{l'Xl}_{000} C^{\mathcal{L}'X\mathcal{L}}_{000}\times \\ &\times \int d\xi_{12}\xi_1^2\xi_2^2 R_{nl}(\xi_1, b) R_{\mathcal{N}\mathcal{L}}(\xi_2, b) R_{n'l'}(\xi_1, b) R_{\mathcal{N}'\mathcal{L}'}(\xi_2, b) \\ &\times Z_0^{(2)}\left(\sqrt{2}\xi_1;\Lambda\right) Z_{0,X}\left(\sqrt{\frac{1}{2}}\xi_1, \sqrt{\frac{3}{2}}\xi_2;\Lambda\right) .\end{aligned}$$

Bare $\mathrm{N^4LO}$ three-nucleon contacts at ^{3}H





[***GP*** *and Navratil, in preparation (2026)*]

Discovery, accelerated

$\mathrm{N^4LO}$ three-nucleon contacts

We won't (over)fit $\mathbf{13}$ new contacts.

Take a subset from an isospin basis (system relevance).

$$\sum_{i=1}^{13} E_i O_i = \sum_{T=1/2}^{3/2} \sum_i h_i^{(T)} D_i^{(T)}$$

where $h_i^{(T)}$ are new LECs.

Two isospin channels: $D_i^{(1/2)}, D_i^{(3/2)}$, defined from the solutions

$$P_T D_i^{(T)} P_T = D_i^{(T)}$$

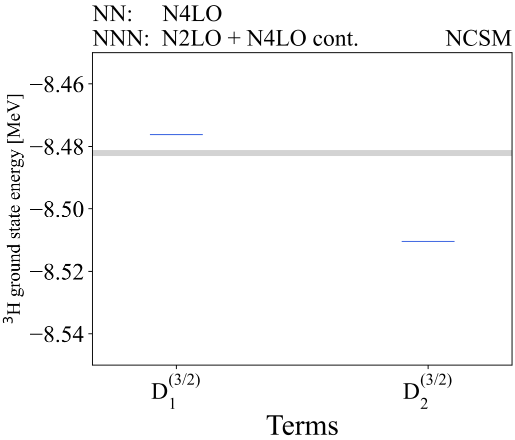
where P_T are isospin projection operators.

N4LO three-nucleon contacts

$$
\begin{array}{c} O_1 \\ O_2 \\ O_3 \\ O_4 \\ O_5 \\ O_6 \\ O_7 \\ O_8 \\ O_9 \\ O_{10} \\ O_{11} \\ O_{12} \\ O_{13} \end{array}
\quad \longrightarrow \quad
\begin{aligned}
D_1^{(1/2)} &= -O_1 + O_2 \\
D_2^{(1/2)} &= -O_7 + O_8 \\
D_i^{(1/2)} &= O_i \ , \quad i = 3,4,5,6 \\
D_7^{(1/2)} &= \frac{1}{2}O_1 + O_7 + O_9 \\
D_8^{(1/2)} &= \frac{1}{2}O_1 - O_7 + O_{10} \\
D_9^{(1/2)} &= \frac{1}{2}O_1 - O_7 + O_{11} \\
D_{10}^{(1/2)} &= \frac{1}{2}O_1 - O_7 + O_{12} \\
D_{11}^{(1/2)} &= \frac{1}{2}O_1 - O_7 + O_{13} \\
D_1^{(3/2)} &= O_2 - \frac{1}{3}(O_4 + O_6) - (O_3 + O_5) - \frac{3}{2}O_9 - \frac{1}{2}O_{10} + O_{13} \\
D_2^{(3/2)} &= -2O_2 + 2(O_3 + O_5) + \frac{2}{3}(O_4 + O_6) + 3O_9 + O_{10} + 3O_{11} + O_{12}
\end{aligned}
$$

Originally derived in [*Alessia Nasoni, BSc Thesis*]

Triton ground state



Local regulator
Bare force

$T = 3/2$ terms shouldn't contribute at a $T = 1/2$ ground state

[***GP** and Navratil, in preparation (2026)*].

Regulator effects

Typically **one** of the **three** possible terms is taken (rest related by permutations)

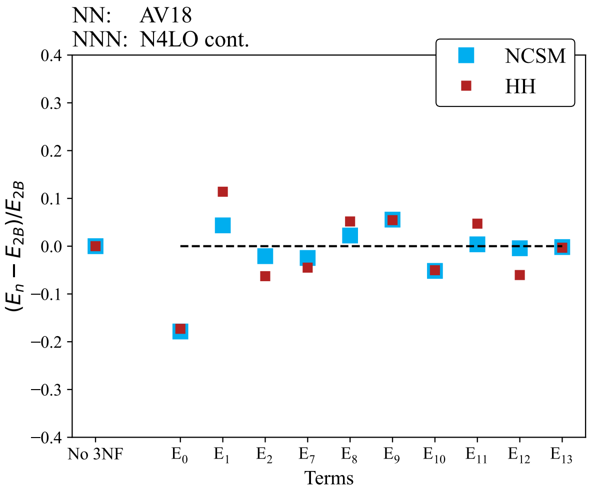
$$\sum_{i\neq j\neq k} W_{ijk} = W_1 + W_2 + W_3$$

For a properly antisymmetrized many-body wavefunction, they are equal so

$$\langle N\alpha JT| \sum_{i\neq j\neq k} W_{ijk} |N'\alpha' JT\rangle = 3\langle N\alpha JT| W_2 |N'\alpha' JT\rangle$$

But the local regulator breaks this.

Comparison with HH



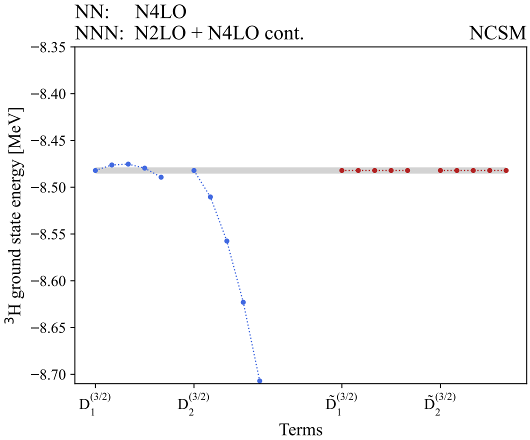
Local regulator
Bare force

With only a 2NF: varying degree of agreement due to regulator effects.

■ *from Luca Girlanda*

[***GP*** *and Navratil, in preparation (2026)*].

$\mathrm{N^4LO}\ T = 3/2$ 3NF contacts



One (easy) option: restore symmetry

$$\tilde{D}_i^{(3/2)} = P_{3/2} D_i^{(3/2)} P_{3/2}$$

$$P_{3/2} = \frac{1}{2} + \frac{1}{6}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_3 + \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3)$$

All combinations

Second option: do it consistently
The projection relations hold for the unregulated terms.

Define a variation of the regulated force:

$$E_2 \sum_{i \neq j \neq k} F(\mathbf{k}_i^2) F(\mathbf{k}_j^2)(-\mathbf{k}_i^2)\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_i \qquad \longrightarrow \qquad E_2 F(\mathbf{k}_2^2) F(\mathbf{k}_3^2) \sum_{i \neq j \neq k} (-\mathbf{k}_i^2)\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_i$$

at the center of mass:

$$\mathbf{k}_1 = -(\mathbf{k}_2 + \mathbf{k}_3)$$

Many repeated terms!

$$\begin{aligned} w_{xy} &= F(\mathbf{k}_2^2) F(\mathbf{k}_3^2) \mathbf{k}_x \cdot \mathbf{k}_y \\ w_{xy;ij} &= F(\mathbf{k}_2^2) F(\mathbf{k}_3^2) (\mathbf{k}_x \cdot \boldsymbol{\sigma}_i)(\mathbf{k}_y \cdot \boldsymbol{\sigma}_j) \\ w_{xy} &= w_{yx} \;, \quad w_{xy;ij} = w_{yx;ji} \;, \quad \ldots \end{aligned}$$

All combinations

Rewrite all terms

$$O_1 = 2\left(w_{22} + w_{23} + w_{33}\right)$$
$$O_2 = [w_{22}(2t_{12} + t_{13} + t_{23}) + w_{33}(t_{12} + 2t_{13} + t_{23}) + 2w_{23}(t_{12} + t_{13})]$$
$$\vdots$$
$$O_{13} = [w_{23;32}(t_{12} + t_{13}) - \left(w_{22;12} + w_{23;12}\right)\left(t_{13} + t_{23}\right) - \left(w_{33;13} + w_{23;31}\right)\left(t_{12} + t_{23}\right)]$$

In terms of **15 distinct terms**

$$\begin{array}{ccc} w_{22}\,, & w_{23}\,, & w_{33}\,, \\ w_{22;23}\,, & w_{22;13}\,, & w_{22;12}\,, \\ w_{33;23}\,, & w_{33;13}\,, & w_{33;12}\,, \\ w_{23;23}\,, & w_{23;13}\,, & w_{23;12}\,, \\ w_{23;32}\,, & w_{23;31}\,, & w_{23;21} \end{array}$$

(similar to the ones showing up at $\mathrm{N^2LO}$)

To be continued . . .

Discovery, accelerated

- Quark-mass dependent 3NFs at light nuclei
  - Scaling of LECs and enhancement
  - Effects on ^{3}H and ^{4}He
  - Tuning strategy

Enhanced 3NFs: D_2 and F_2

Identified by [*Cirigliano et al, PRL (2025)*]:

$$W_{D_2} = \sum_{i\neq j\neq k} \frac{9g_A^2 D_2 m_\pi^3}{128\pi F_\pi^4} \frac{\left(d_2^S + d_2^T \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right)}{d_2^S - 3d_2^T} \mathcal{I}\left(\frac{q_k^2}{4m_\pi^2}\right) ,$$

$$W_{F_2} = -\sum_{i\neq j\neq k} \frac{15g_A^2 m_\pi^3}{16\pi F_\pi^4} \left(f_2^S + f_2^T \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right) \mathcal{J}\left(\frac{q_k^2}{4m_\pi^2}\right) .$$

Enhanced based on renormalization group scaling

$$\mathrm{N^5LO} \quad \longrightarrow \quad \mathrm{N^3LO}$$

See also [*Vernik et al, 212.20454 (2026)*] and **U. Vernik's talk**

Enhanced 3NFs: The scaling of the LECs

In the Weinberg counting, the LECs can be counted as
[*Friar, FBS (1997)*], [*Epelbaum et al, arXiv:2512.14117 (2025)*]

$$X = c_X \left(\frac{\bar{N}(\cdots)N}{F_\pi^2\Lambda_\chi}\right)^l \left(\frac{\boldsymbol{\pi}}{F_\pi}\right)^m \left(\frac{\partial^\mu, m_\pi}{\Lambda_\chi}\right)^n F_\pi^2\Lambda_\chi^2$$

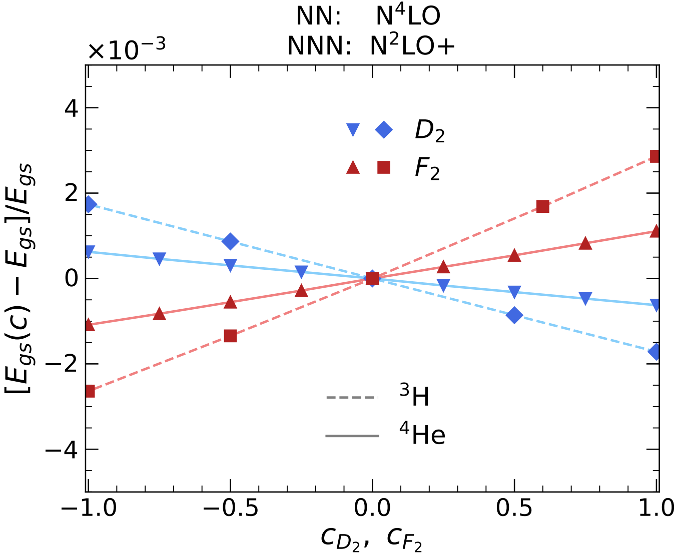
For the new terms, this is

$$D_2 = \frac{c_{D_2}}{F_\pi^2\Lambda_\chi^2}\,, \quad F_2 = \frac{c_{F_2}}{F_\pi^2\Lambda_\chi^2}$$

$$E = \frac{c_E}{F_\pi^4\Lambda_\chi}$$

$$(\text{comparison with N}^2\text{LO})$$

Enhanced 3NFs: Weinberg counting at ^{3}H and ^{4}He



Enhanced 3NFs: The scaling of the LECs

In [*Cirigliano et al, PRL (2025)*], the LECs are counted as,

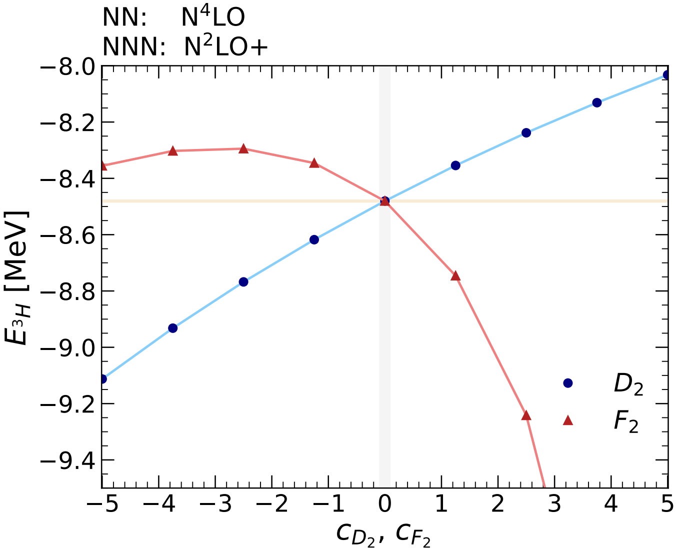
$$D_2 = \frac{c_{D_2}}{5F_\pi^4}\,, \quad F_2 = \frac{c_{F_2}}{5F_\pi^4}$$

$$D_2 = \frac{c_{D_2}}{F_\pi^2\Lambda_\chi^2}\,, \quad F_2 = \frac{c_{F_2}}{F_\pi^2\Lambda_\chi^2}$$

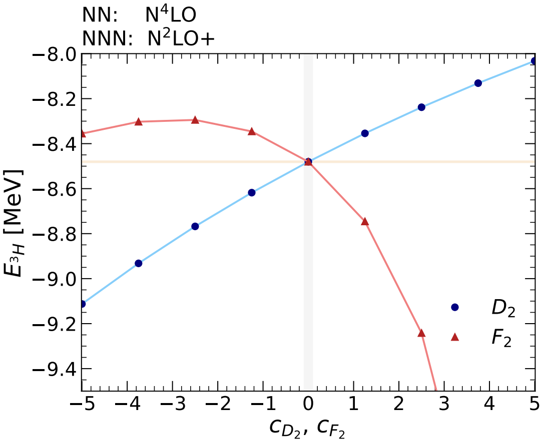
For $F_\pi = 92.4\ \text{MeV}$ and $\Lambda_\chi = 700\ \text{MeV}$.

$$\times\frac{1}{5}\left(\frac{\Lambda_\chi}{F_\pi}\right)^2 \approx 10 \qquad \text{enhancement}$$

Enhanced 3NFs: Enhanced counting at ^{3}H



Enhanced 3NFs: Enhanced counting at ^{3}H



Tuning strategy:

- Keep c_D fixed (fit to GT ME)
- For each c_{F_2} and c_{D_2}, fix c_E at ^{3}H
- Fix c_{D_2} and c_{F_2} at p-shell nuclei

Summary & Next steps

Summary

- Subleading $\mathrm{N^4LO}$ contacts
  - On our way towards a consistent implementation of the subleading 3NF contacts at the $\mathrm{N^4LO}$.
  - Two ways to define local $T = 3/2$ contributions can be tested
- Enhanced 3NFs
  - Effects on the triton seem consistent with mean-field nuclear matter calculations
  - Calculations for p-shell underway ...

Some Next steps

1. Explore importance of subleading contacts for neutron rich-er nuclei
2. Explore subleading contacts in neutron drops (regulator effects)
3. Tune enhanced 3NFs & explore effects in light nuclei

Thank you.

N4LO 3NF: Partial Waves

$$O_1 = \sum_{i\neq j\neq k} \left(-k_i^2\right) \ , \quad O_2 = \sum_{i\neq j\neq k} \left(-k_i^2\right) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j \ ,$$

$$V_{1,2} = \sum_{i\neq j\neq k} \left(E_1 + E_2 \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j\right) \left[Z_0''(r_{ij}) + 2\frac{Z_0'(r_{ij})}{r_{ij}}\right] Z_0(r_{ik})$$

Similar to c_E. Partial wave decomposition

$$E_1 W_1 + E_2 W_2 = \delta_{ss'} \left[E_1 \delta_{tt'} + E_2 \hat{t}\hat{t}'(-1)^{t+t'+T+\frac{1}{2}} \begin{Bmatrix} t & t' & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} t & t' & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix}\right] \times$$
$$\times \hat{j}\hat{j}'\hat{\mathcal{J}}\hat{\mathcal{J}}'\hat{l}'\hat{\mathcal{L}}'(-1)^{J-\frac{1}{2}\mathcal{J}'-\mathcal{J}+l+\mathcal{L}+s} \times$$
$$\times \sum_X (-1)^X \hat{X}^2 \begin{Bmatrix} l & s & j \\ j' & X & l' \end{Bmatrix} \begin{Bmatrix} X & j & j' \\ J & \mathcal{J}' & \mathcal{J} \end{Bmatrix} \begin{Bmatrix} \mathcal{L}' & 1/2 & \mathcal{J}' \\ \mathcal{J} & X & \mathcal{L} \end{Bmatrix} C^{l'Xl}_{000} C^{\mathcal{L}'X\mathcal{L}}_{000} \times$$
$$\times \int d\xi_{12} \xi_1^2 \xi_2^2 R_{nl}(\xi_1, b) R_{\mathcal{N}\mathcal{L}}(\xi_2, b) R_{n'l'}(\xi_1, b) R_{\mathcal{N}'\mathcal{L}'}(\xi_2, b)$$
$$\times Z_0^{(2)}\left(\sqrt{2}\xi_1; \Lambda\right) Z_{0,X}\left(\sqrt{\frac{1}{2}}\xi_1, \sqrt{\frac{3}{2}}\xi_2; \Lambda\right) .$$

Discovery, accelerated

N4LO 3NF: Partial Waves

$$O_3 + O_5 = \tilde{O}_5 = \sum_{i \neq j \neq k} \left(-3\mathbf{k}_i \cdot \boldsymbol{\sigma}_i \mathbf{k}_i \cdot \boldsymbol{\sigma}_j\right) \;, \quad O_4 + O_6 = \tilde{O}_6 = \sum_{i \neq j \neq k} \left(-3\mathbf{k}_i \cdot \boldsymbol{\sigma}_i \mathbf{k}_i \cdot \boldsymbol{\sigma}_j\right) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j \;.$$

Similar to c_D. Partial wave decomposition

$$E_5 \tilde{W}_5 + E_6 \tilde{W}_6 = -36 \delta_{ss'} \left[E_5 \delta_{tt'} + E_6 3 \hat{t} \hat{t}' (-1)^{t+t'+T+\frac{1}{2}} \begin{Bmatrix} t & t' & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} t & t' & 1 \\ \frac{1}{2} & \frac{1}{2} & T \end{Bmatrix} \right] \hat{j} \hat{j}' \hat{\mathcal{J}} \hat{\mathcal{J}}' \hat{s} \hat{s}'$$
$$\times (-1)^{J-\mathcal{J}+s+j'} \begin{Bmatrix} s & s' & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \hat{l}' \hat{\mathcal{L}}' \sum_{K=0,2} \hat{K} (-1)^{K/2} \sum_{VX} (-1)^V \hat{V} \hat{X}^2 C^{11K}_{000} C^{Vl'l}_{000} C^{XKV}_{000} C^{X\mathcal{L}'\mathcal{L}}_{000}$$
$$\times \sum_Z \hat{Z}^2 \begin{Bmatrix} l & s & j \\ l' & s' & j' \\ V & 1 & Z \end{Bmatrix} \begin{Bmatrix} \mathcal{L} & \frac{1}{2} & \mathcal{J} \\ \mathcal{L}' & \frac{1}{2} & \mathcal{J}' \\ X & 1 & Z \end{Bmatrix} \begin{Bmatrix} j & j' & Z \\ \mathcal{J}' & \mathcal{J} & J \end{Bmatrix} \begin{Bmatrix} V & 1 & Z \\ 1 & X & K \end{Bmatrix}$$
$$\times \int d\xi_{12} \xi_1^2 \xi_2^2 R_{nl}(\xi_1, b) R_{\mathcal{N}\mathcal{L}}(\xi_2, b) R_{n'l'}(\xi_1, b) R_{\mathcal{N}'\mathcal{L}'}(\xi_2, b) f_K\left(\sqrt{2}\xi_1; \Lambda\right) Z_{0,X}\left(\sqrt{\frac{1}{2}}\xi_1, \sqrt{\frac{3}{2}}\xi_2; \Lambda\right)$$

N4LO 3NF: Partial Waves

$$O_9 = \sum_{i \neq j \neq k} \left(-\mathbf{k}_i \cdot \boldsymbol{\sigma}_i \mathbf{k}_j \cdot \boldsymbol{\sigma}_j \right) \ , \quad O_{10} = \sum_{i \neq j \neq k} \left(-\mathbf{k}_i \cdot \boldsymbol{\sigma}_i \mathbf{k}_j \cdot \boldsymbol{\sigma}_j \right) \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$$

Similar to c_1. Partial wave decomposition

$$E_9 W_9 + E_{10} W_{10} = -6 \left[E_5 \delta_{tt'} + E_6 \langle \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \rangle \right] (-1)^{s+j'+J-\mathcal{J}} \hat{l}' \hat{\mathcal{L}}' \hat{s} \hat{s}' \hat{j} \hat{j}' \hat{\mathcal{J}} \hat{\mathcal{J}}' \left\{ \begin{array}{ccc} 1 & s & s' \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right\}$$

$$\times \sum_{XVRY} \sum_{K_3=0}^{1} (-1)^Y \hat{X}^2 \hat{V} \hat{R} \hat{Y} \widehat{1-K_3} \left[\binom{3}{2K_3} \right]^{\frac{1}{2}} C^{X K_3 Y}_{000} C^{X 1-K_3 R}_{000} C^{Y 1 V}_{000} C^{l' V l}_{000} C^{\mathcal{L}' R \mathcal{L}}_{000} \times$$

$$\times \left\{ \begin{array}{ccc} Y & X & K_3 \\ 1-K_3 & 1 & R \end{array} \right\} \left\{ \begin{array}{ccc} Y & j' & j \\ J & \mathcal{J} & \mathcal{J}' \end{array} \right\} \left\{ \begin{array}{ccc} j & l & s \\ j' & l' & s' \\ Y & V & 1 \end{array} \right\} \left\{ \begin{array}{ccc} \mathcal{J} & \mathcal{L} & \frac{1}{2} \\ \mathcal{J}' & \mathcal{L}' & \frac{1}{2} \\ Y & R & 1 \end{array} \right\}$$

$$\times \int d\xi_{12} \xi_1^2 \xi_2^2 R_{nl}(\xi_1) R_{\mathcal{N}\mathcal{L}}(\xi_2) R_{n'l'}(\xi_1) R_{\mathcal{N}'\mathcal{L}'}(\xi_2) f_1\left(\sqrt{2}\xi_1\right) \left(\sqrt{\frac{1}{2}}\xi_1\right)^{K_3} \left(\sqrt{\frac{3}{2}}\xi_2\right)^{1-K_3}$$

$$\times f_{1,X}\left(\sqrt{\frac{1}{2}}\xi_1, \sqrt{\frac{3}{2}}\xi_2; \Lambda\right)$$

N4LO 3NF: Partial Waves

$$O_{11}=\sum_{i\neq j\neq k}\left(-\mathbf{k}_i\cdot\boldsymbol{\sigma}_j\mathbf{k}_j\cdot\boldsymbol{\sigma}_i\right)\;,\quad O_{12}=\sum_{i\neq j\neq k}\left(-\mathbf{k}_i\cdot\boldsymbol{\sigma}_j\mathbf{k}_j\cdot\boldsymbol{\sigma}_i\right)\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\;,\quad O_{13}=\sum_{i\neq j\neq k}\left(-\mathbf{k}_i\cdot\boldsymbol{\sigma}_j\mathbf{k}_j\cdot\boldsymbol{\sigma}_i\right)\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k$$

Partial wave decomposition

$$\begin{aligned}
E_{11}W_{11}+E_{12}W_{12}+E_{13}W_{13} &= -6\left[E_{11}\delta_{tt'}+E_{12}\left\langle\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3\right\rangle+E_{13}\left\langle\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_3\right\rangle\right]\\
&\times\begin{Bmatrix}1 & s & s'\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}\hat{l}'\hat{\mathcal{L}}'\hat{s}\hat{s}'\hat{j}\hat{j}'\hat{\mathcal{J}}\hat{\mathcal{J}}'(-1)^{s+j'+J-\mathcal{J}}\sum_{VR}\hat{V}\hat{R}C^{l'Vl}_{000}C^{\mathcal{L}'R\mathcal{L}}_{000}\sum_{Y}\hat{Y}C^{Y1V}_{000}\times\\
&\times\sum_{G}(-1)^{G}\hat{G}^2\begin{Bmatrix}1 & V & G\\ 1 & R & Y\end{Bmatrix}\begin{Bmatrix}G & j' & j\\ J & \mathcal{J} & \mathcal{J}'\end{Bmatrix}\begin{Bmatrix}j & l & s\\ j' & l' & s'\\ G & V & 1\end{Bmatrix}\begin{Bmatrix}\mathcal{J} & \mathcal{L} & \frac{1}{2}\\ \mathcal{J}' & \mathcal{L}' & \frac{1}{2}\\ G & R & 1\end{Bmatrix}\times\\
&\times\sum_{K_3=0}^{1}\left[\binom{3}{2K_3}\right]^{\frac{1}{2}}\widehat{1-K_3}\sum_{X}\hat{X}^2C^{XK_3Y}_{000}C^{X1-K_3R}_{000}\begin{Bmatrix}Y & X & K_3\\ 1-K_3 & 1 & R\end{Bmatrix}\times\\
&\times\int d\xi_{12}\xi_1^2\xi_2^2R_{nl}\left(\xi_1\right)R_{\mathcal{N}\mathcal{L}}\left(\xi_2\right)R_{n'l'}\left(\xi_1\right)R_{\mathcal{N}'\mathcal{L}'}\left(\xi_2\right)f_1\left(\sqrt{2}\xi_1\right)\left(\sqrt{\frac{1}{2}}\xi_1\right)^{K_3}\left(\sqrt{\frac{3}{2}}\xi_2\right)^{1-K_3}\\
&\times f_{1,X}\left(\sqrt{\frac{1}{2}}\xi_1,\sqrt{\frac{3}{2}}\xi_2;\Lambda\right)
\end{aligned}$$