A Coincidence Algebra Bundle for Decay Quivers

"An Algebraic Approach to Gamma-ray Spectroscopy"

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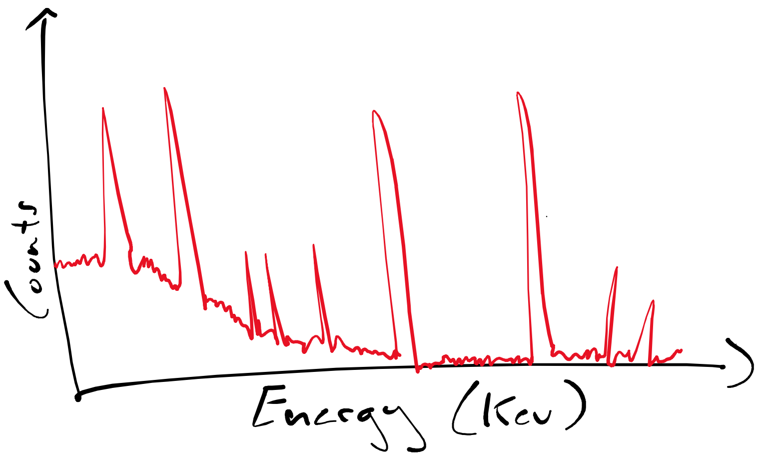
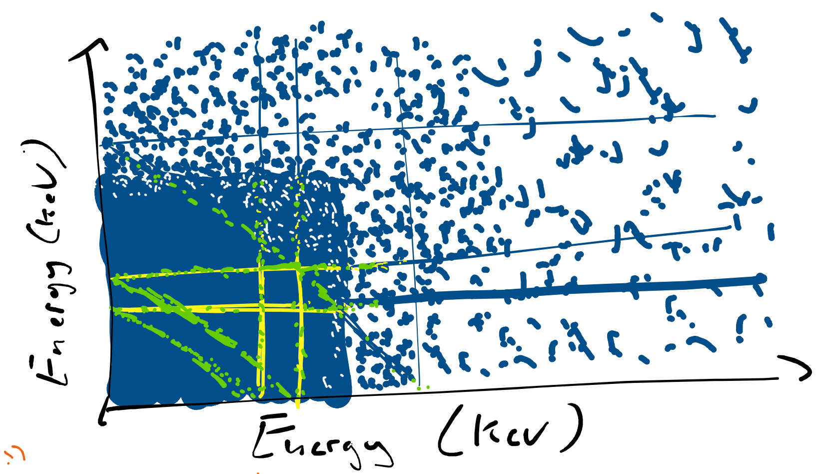
What is the general Problem of gamma-ray Spectroscopy?

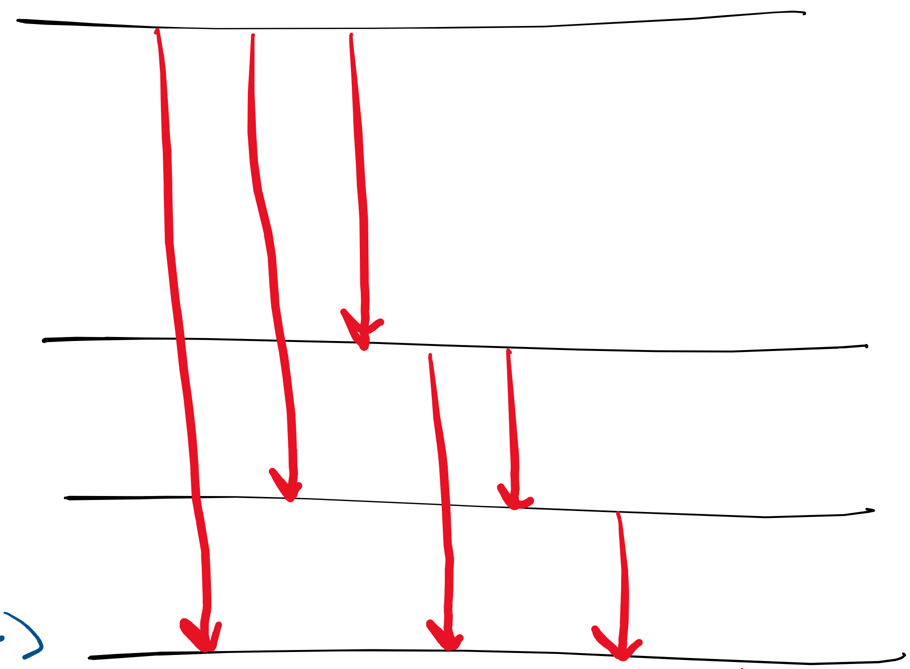
(For Nuclear Structure)

Detected Events ⇒ Nuclear Level Schemes





"Detected Coincidence Probabilities"



"Branching Ratios/
Transition Probabilities"

$$\underline{\text{A Quiver}} \quad Q = \Big(Q_0, Q_1, \underbrace{s, t}_{\text{Source \& Target Maps}}\Big)$$

"Set of Vertices"

$(\hat{\varepsilon}_0, \hat{\varepsilon}_1, \hat{\varepsilon}_2)$

"Set of Arrows"

$(\hat{p}_{21}, \hat{p}_{20}, \hat{p}_{10})$

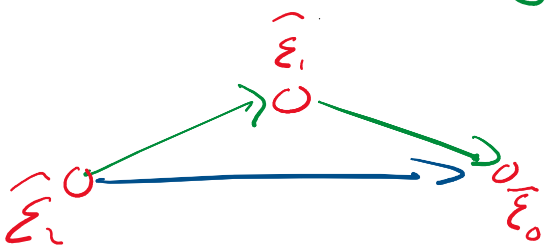
$$s(\hat{p}_{21}) = \hat{\varepsilon}_2$$

$$t(\hat{p}_{21}) = \hat{\varepsilon}_1$$

$$\begin{tikzcd} & \hat{\varepsilon}_1 \arrow[dr, "\hat{p}_{10}"] & \\ \hat{\varepsilon}_2 \arrow[ur, "\hat{p}_{21}"] \arrow[rr, "\hat{p}_{20}"'] & & \hat{\varepsilon}_0 \end{tikzcd}$$

Path Algebra of a Quiver

- Path of length ℓ = ℓ connected arrows



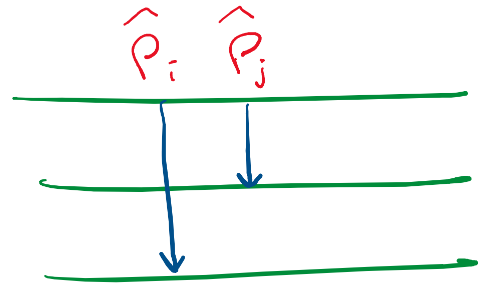
"$\widehat{P_{21}P_{10}}$ is the path of length 2 from $\hat{\mathcal{E}}_2$ to $\hat{\mathcal{E}}_0$"

- $\{\text{All Paths}\} \Rightarrow$ Basis of Vector Space kQ
- Path Algebra: $\boxed{\hat{P}_{21} \cdot \hat{P}_{10} = \widehat{P_{21}P_{10}}}$ "Path Concatenation"
- $\hat{\mathcal{E}}_i$ are Stationary Paths with $\ell = 0$: $\boxed{\hat{\mathcal{E}}_2 \cdot \hat{P}_{21} = \hat{P}_{21}}$

Comparing Paths

- "Source Bilinear Form":

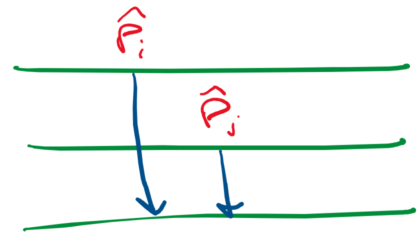
$$\langle \hat{p}_i, \hat{p}_j \rangle_s = \delta_{s(\hat{p}_i)\, s(\hat{p}_j)}$$



"Diverging Paths"

- "Target Bilinear Form":

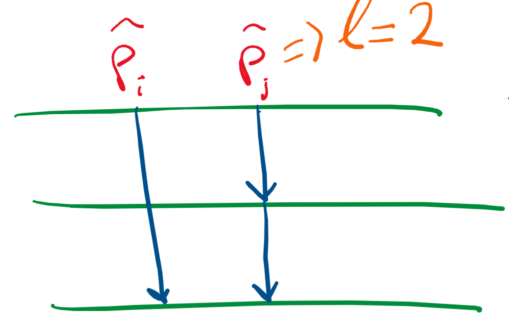
$$\langle \hat{p}_i, \hat{p}_j \rangle_t = \delta_{t(\hat{p}_i)\, t(\hat{p}_j)}$$



"Converging Paths"

- "Path Bilinear Form":

$$\langle \hat{p}_i, \hat{p}_j \rangle_p = \delta_{t(\hat{p}_i)\, t(\hat{p}_j)}\, \delta_{s(\hat{p}_i)\, s(\hat{p}_j)}$$



"Equivalent Paths"

Projectors

- "Target Vertex Projector":

$$V_t(d) = \sum_{\hat{\epsilon} \in P_0} \langle \hat{\epsilon}, d \rangle_t \hat{\epsilon}$$

"Sums over converging Paths onto vertices"

$$d = X_{21} \hat{P}_{21} + X_{31} \hat{P}_{31}$$

$$V_t(d) = (X_{21} + X_{31}) \hat{\epsilon}_1$$

- "Source Vertex Projector":

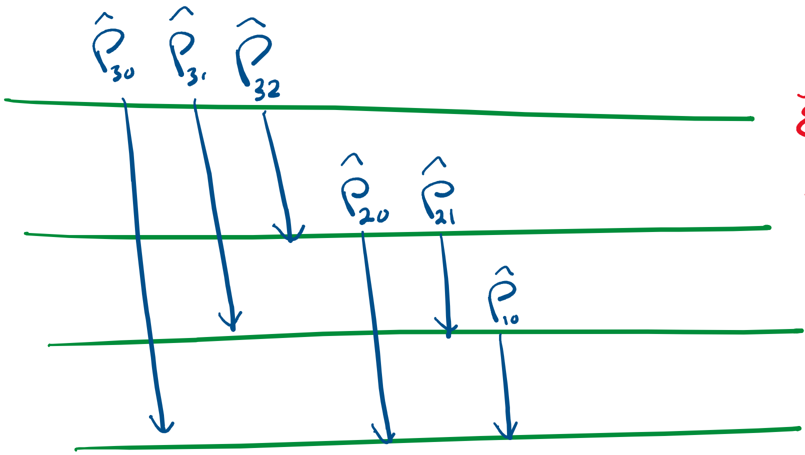
$$V_s(d) = \sum_{\hat{\epsilon} \in P_0} \langle \hat{\epsilon}, d \rangle_s \hat{\epsilon}$$

"Sums over diverging Paths onto vertices"

$$d = X_{32} \hat{P}_{32} + X_{31} \hat{P}_{31}$$

$$V_s(d) = (X_{32} + X_{31}) \hat{\epsilon}_3$$

Decay Quivers:



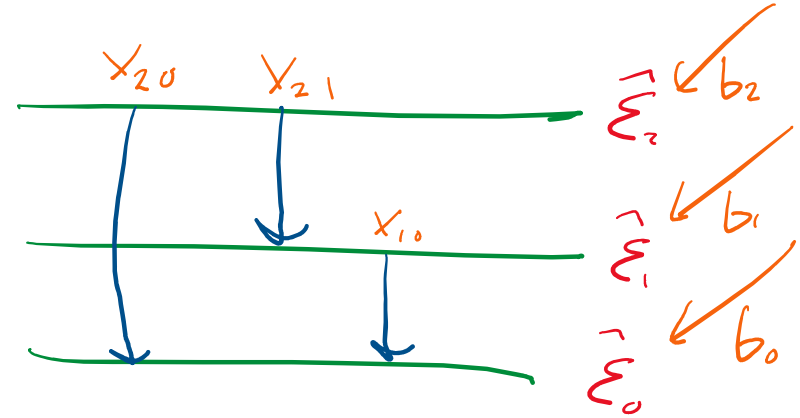
$\hat{\mathcal{E}}_3$ -> 3 arrows
$\hat{\mathcal{E}}_2$ -> 2 arrows
$\hat{\mathcal{E}}_1$ -> 1 arrow
$\hat{\mathcal{E}}_0$ -> 0 arrows

"A Decay Quiver D is a Quiver whose Vertex Set is Totally Ordered by a vertices number of diverging arrows."

$$\hat{\mathcal{E}}_i \preceq \hat{\mathcal{E}}_j \iff N(\hat{\mathcal{E}}_i) \leq N(\hat{\mathcal{E}}_j)$$

- A Decay Vector $d \in KD$ is a vector whose Coefficients are Branching Ratios/Transition Probabilities.

Decay Vectors



- A vector $d \in KD$:

$$d = b_0\hat{\varepsilon}_0 + b_1\hat{\varepsilon}_1 + b_2\hat{\varepsilon}_2$$
$$+ X_{20}\hat{P}_{20} + X_{21}\hat{P}_{21} + X_{10}\hat{P}_{10}$$

- Path Concatenation $\Rightarrow$ Cascade Terms

$$d \cdot d = b_1 X_{10}\hat{P}_{10} + b_2\left(X_{21}\hat{P}_{21} + X_{20}\hat{P}_{20}\right) + X_{21}X_{10}\widehat{P_{21}P_{10}}$$

Emission Probabilities in PA

- Decompose decay vector: $d = b + \tau$ ← Branching Ratios $(\hat{\varepsilon}_i)$; ← Transition Probabilities $(\hat{p}_i)$
- Power expand to create cascades:

$$\bar{\tau}^n = \sum_{k=0}^{n} \tau^k$$

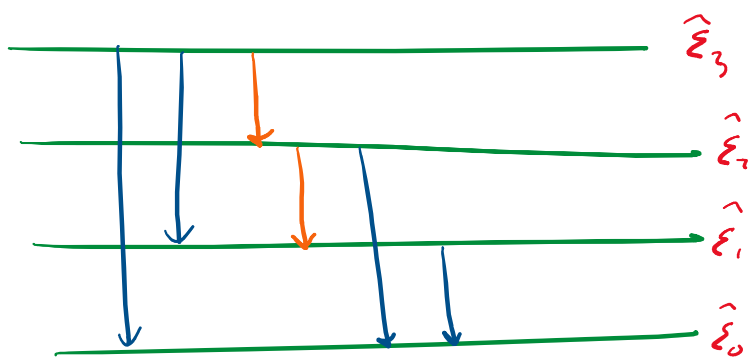
* PA multiplication!

- Emission Probability of path $\hat{p}_i$:

$$\boxed{P(\hat{p}_i) = \left\langle \mathcal{N}_t (b \cdot \bar{\tau}^n) \cdot t, \hat{p}_i \right\rangle_p}$$

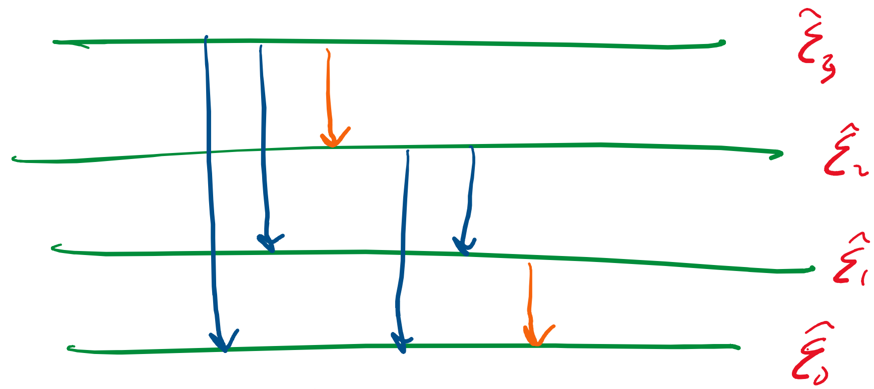
- Path Algebra only creates cascade terms.
- How can we allow for general Coincidence terms?

Path Algebra



$$\hat{P}_{32} \cdot \hat{P}_{21} = \widehat{P_{32} P_{21}}$$

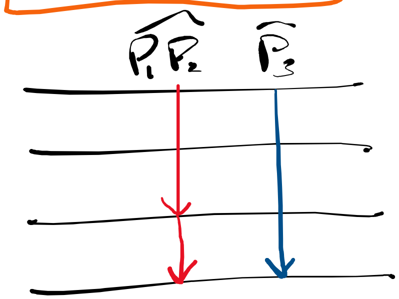
Coincidence Algebra?



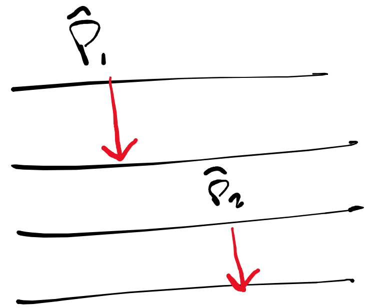
$$\hat{P}_{32} \bullet_c \hat{P}_{10} = ?$$

Basis Vectors In The Coincidence Space

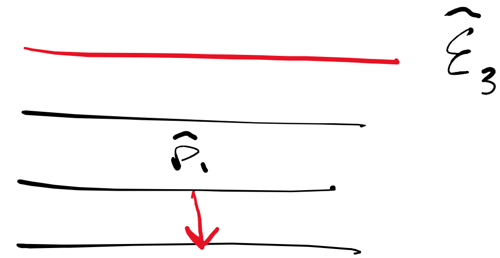
Paths: $\oplus$ **Tensor Products:**



$\widehat{P_1P_2}$, $\hat{P}_3$



$\hat{P}_1 \otimes \hat{P}_2$



$\hat{\mathcal{E}}_3 \otimes \hat{P}_1$

$C = \text{Span}\left\{t = \bigotimes_{i=1}^{|c|} \hat{P}_i \,\middle|\, \hat{P}_i \in c, \forall c \in O \cup P\right\}$ "Coincidence Vector Space"

where, $O = \left\{u \in \mathbb{P}(P) \,\middle|\, |u| \geq 2,\ \forall \hat{P}_1, \hat{P}_2 \in u,\ S(\hat{P}_1) < t(\hat{P}_2) \vee S(\hat{P}_2) > t(\hat{P}_1)\right\}$ "Coincidence Set"

Coincidence Multiplication

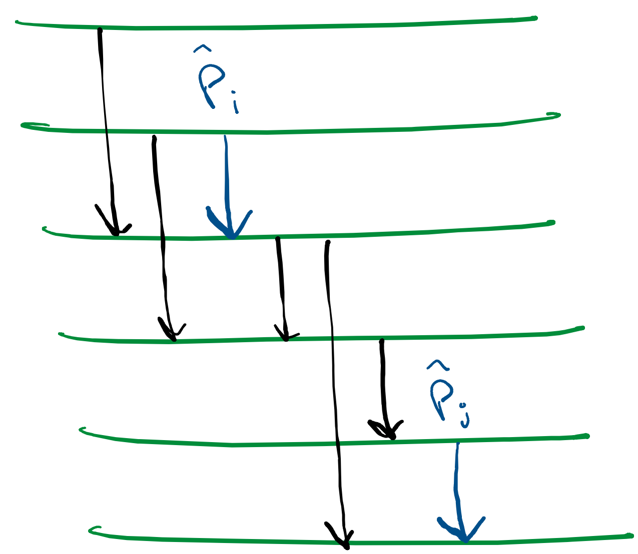
- Multiplication for non-connected paths?

$$\hat{P}_i \cdot_c \hat{P}_j \neq \hat{P}_i \otimes \hat{P}_j$$

(In General)

- Must reproduce Coincidence probabilities!

$$\hat{P}_i \cdot_c \hat{P}_j = \left(\text{“Probability of } \hat{P}_i \text{ leading to } \hat{P}_j\text{”} \right) \hat{P}_i \otimes \hat{P}_j$$



$$\hat{P}_i \cdot_c \hat{P}_j = ?$$

Scalar Connection

- What connects $\hat{P}_{64} \to \hat{P}_{10}$?

- Possible Paths:

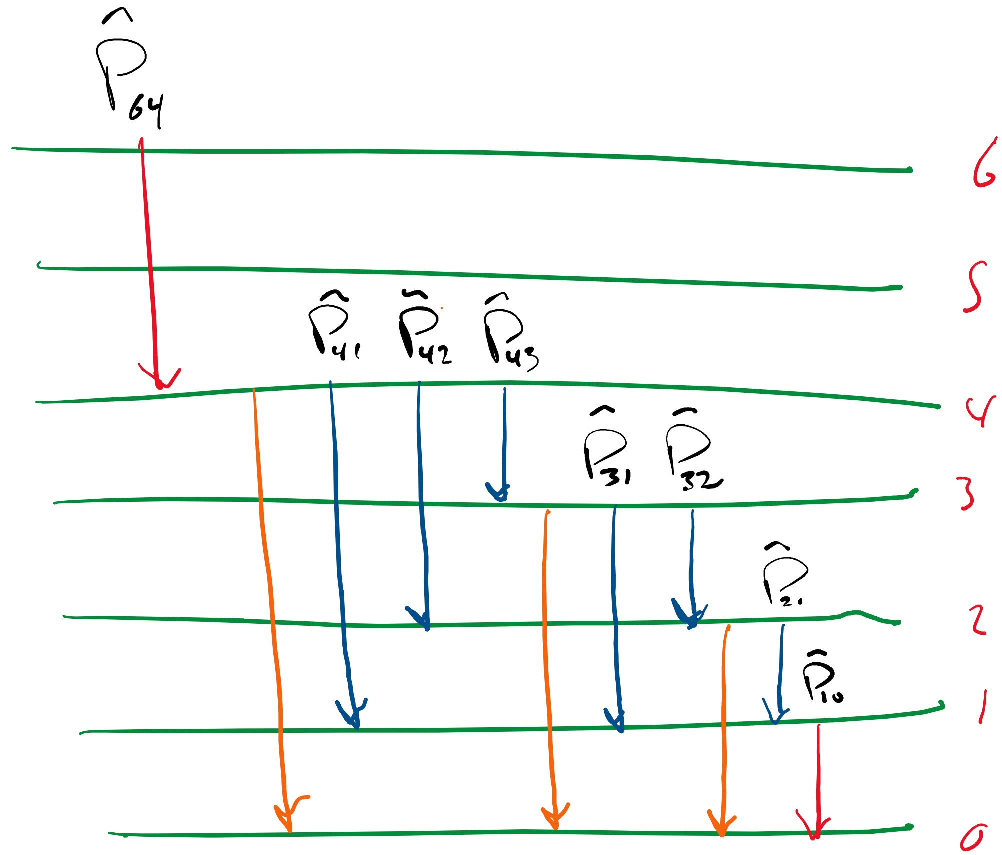
  $\hat{P}_{41}$

  $\overline{\hat{P}_{42}\hat{P}_{21}}$

  $\overbrace{\hat{P}_{43}\hat{P}_{32}\hat{P}_{21}}$

  * Depends on transition Vector!

- Scalar Connection $C_d(\hat{p}_i, \hat{p}_j)$ sums over these connecting paths.



$$C_d = \left\langle V_t\left(\epsilon_{t(\hat{p}_i)}, \overline{\tau}^n\right), \zeta_{s(\hat{p}_j)} \right\rangle_p \Rightarrow$$

"Probability $t(\hat{p}_i)$ decays to the $S(\hat{p}_j)$"

Coincidence Algebra Bundle $\mathcal{A} \xrightarrow{\Phi} kD$

- For each $d \in kD$, an algebra can be defined with $C_d(\hat{p}_i, \hat{p}_j)$ as its structure constants.

$$\hat{P}_i \cdot_d \hat{P}_j = \begin{cases} C_d(\hat{p}_i,\hat{p}_j)\, \hat{p}_i \otimes \hat{p}_j & , S(\hat{p}_j) \prec t(\hat{p}_i) \\ \widehat{p_i p_j} & , S(\hat{p}_j) = t(\hat{p}_i) \\ 0 & , \text{else} \end{cases}$$

- Emission Probability Vector $\Gamma_1(d)$

$$\Gamma_1(d) = \tilde{b} \cdot_d \tilde{\tau}$$

Transitions $\longrightarrow$ Basis Vectors

Probabilities $\longrightarrow$ Coefficients

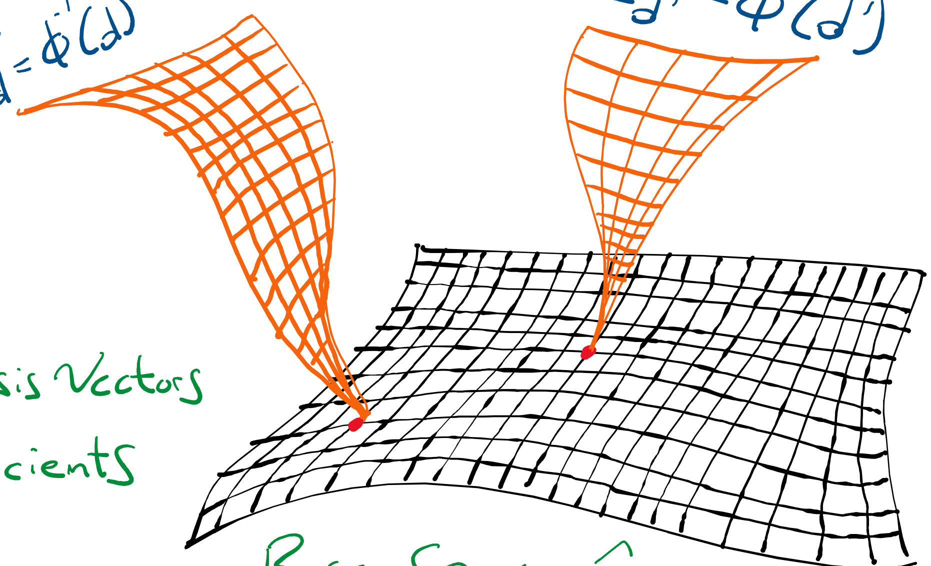
- Coincidence Probability Vector $\Gamma_2(d)$

$$\Gamma_2(d) = \tilde{b} \cdot_d \tilde{\tau} \cdot_d \tilde{\tau}$$

Coincidences $\longrightarrow$ Basis Vectors

Coincidence Algebra fibres

$C_d = \Phi^{-1}(d)$ $\quad$ $C_{d'} = \Phi^{-1}(d')$



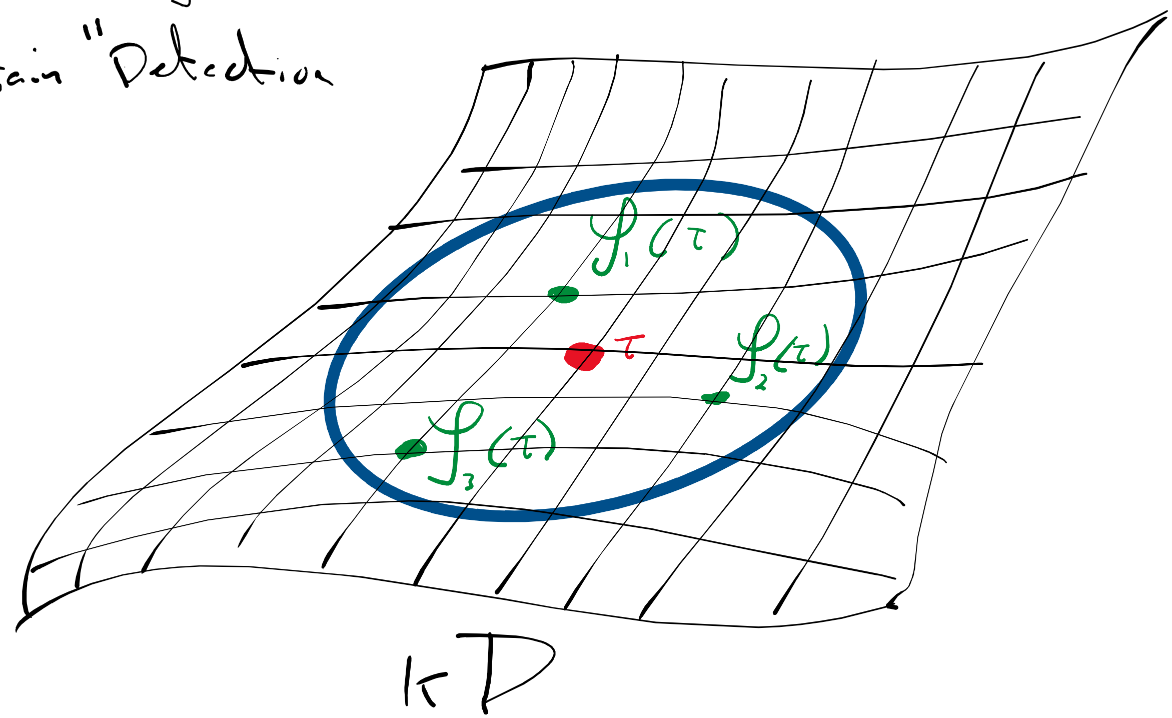
Base Space

kQ

"Detection" Vectors in the Base Space

- A given decay quiver has one decay vector $d = b + \tau$
- There exist other "probability vectors" in kD
- These vectors may contain "Detection Probabilities"

$$\tau = X_{21}\hat{P}_{21} + X_{10}\hat{P}_{10}$$



$$\varphi_{e^P}(\tau) = e^P_{21} X_{21}\hat{P}_{21} + e^P_{10} X_{10}\hat{P}_{10}$$

↳ "Peak efficiencies for each transition"

Detection Maps

Full Energy Detection

$$X_{ij} \longrightarrow e^{P}_{ij} X_{ij}$$

Compton Scattering

$$X_{ij} \longrightarrow e^{T}_{ij} X_{ij}$$

A γ-rays detection can be "summed-out" due to other γ-rays being in the same detector

Summing-Out

$$X_{ij} \longrightarrow \underbrace{\left(1 - \frac{e^{T}_{ij}}{N}\right)} X_{ij}$$

"Probability X_{ij} does not deposit any energy in a given detector"

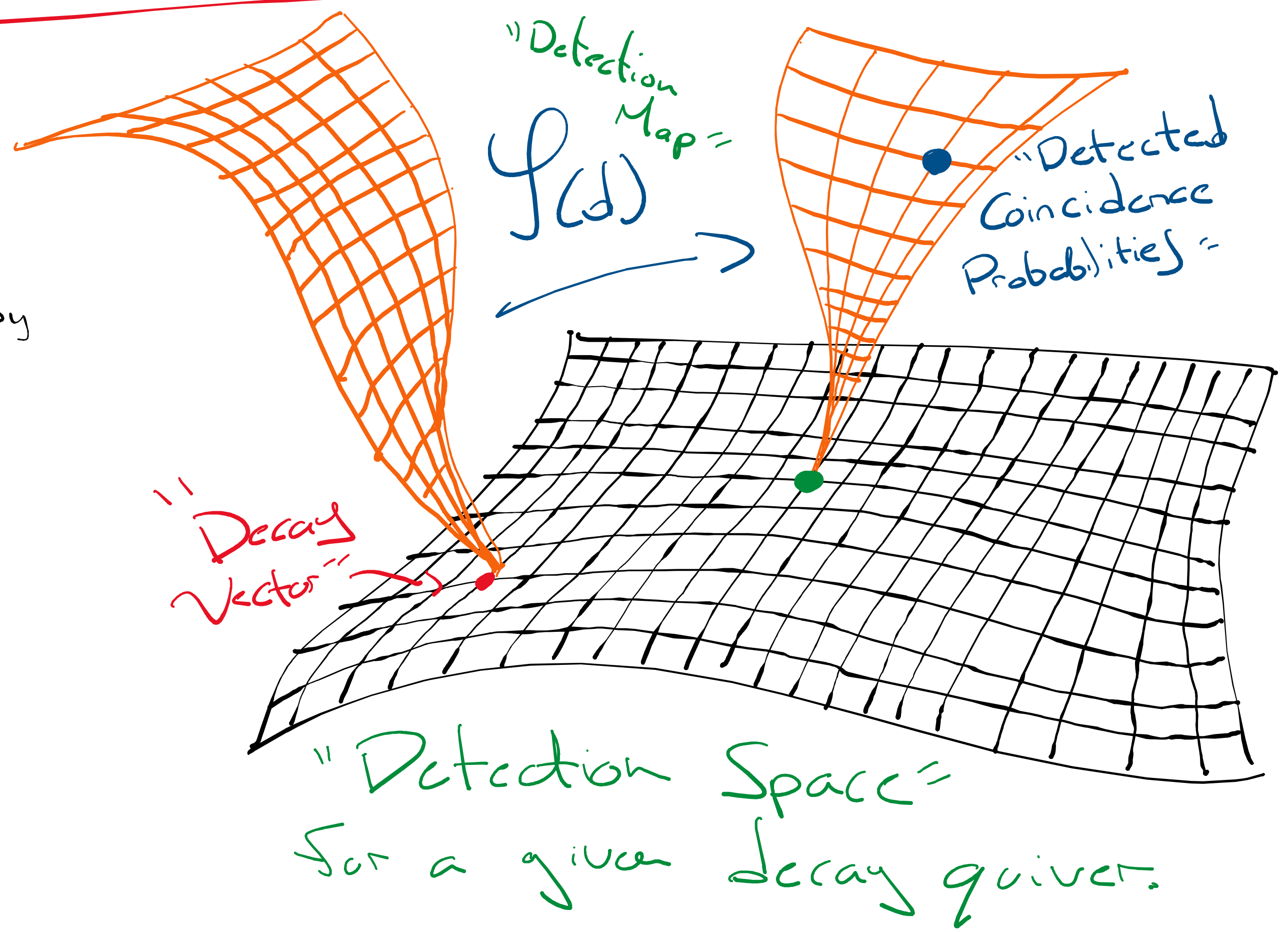
(N detectors, Isotropic emission)

The Landscape of Detection & Coincidence Probabilities

- The CA-Bundle may provide the correct landscape to frame the general problem of Gamma-ray Spectroscopy for nuclear structure.

- Investigation into the deeper structure of the CA-Bundle may lead to determinability theorems for nuclear level schemes.

- At the very least provides the structure to handle the combinatorics involved in Gamma-ray Spectroscopy



Thank You

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[1] I. Assem, A. Skowronski, and D. Simson, *Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory*, London Mathematical Society Student Texts (Cambridge University Press, 2006).

[2] T. M. Semkow, G. Mehmood, P. P. Parekh, and M. Virgil, Coincidence summing in gamma-ray spectroscopy, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment **290**, 437 (1990).