Mass Degeneracies in Extended Higgs Sectors

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Vancouver is well known for degeneracies (e.g. the Sedin twins)
Outline

1. Flashback to 2012: Can mass-degenerate scalars of the 2HDM explain the $h \rightarrow \gamma \gamma$ anomaly (which has since disappeared)?

2. The simplest model of scalar mass degeneracy—the charged Higgs boson

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4. A warmup: Neutral scalar mass degeneracies in the 2HDM
   - Scalar mass degeneracies in the inert doublet model (IDM)
   - Extending the IDM results to more general 2HDMs

5. New features of mass degenerate scalars in the 3HDM
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   - The Ivanov-Silva model (and the importance of CP4)

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Flashback to 2012: Can Mass-degenerate scalars explain the $h \rightarrow \gamma\gamma$ anomaly?

After the initial discovery of the Higgs boson in 2012, it appeared that the signal strength for $h \rightarrow \gamma\gamma$ was significantly enhanced above Standard Model (SM) expectations.

My collaborators and I proposed to explain this anomaly under the assumption that the observed Higgs state at 125 GeV was in fact a pair of mass degenerate scalars.*

We considered the Type-I and Type-II two-Higgs doublet model (2HDM), and explored various possibilities for mass degeneracy and their phenomenological consequences.

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Constraints on 2HDM-I and 2HDM-II parameter scans

We varied the model parameters under the assumptions that

- The scalar potential is bounded from below and the quartic Higgs couplings lie below their unitarity limits;
- \( m_{h^0} \) is taken to be the observed mass;
- the couplings of \( h^0 \) to \( VV (V = W \text{ or } Z) \) are within 20% of their SM-values;
- Precision electroweak constraints (contributions to \( S, T \) and \( U \)) are satisfied;

subject to the constraints imposed by the impact of virtual Higgs effects in \( B \)-physics.

In our parameter scans, we sometimes omitted the \( B \)-physics constraints, since the effects of virtual Higgs exchange could be canceled by contributions from other new BSM physics. Such scans were called “unconstrained.”
We defined

\[ R_f^H = \frac{\sigma(pp \to H)_{2HDM} \times BR(H \to f)_{2HDM}}{\sigma(pp \to h_{SM}) \times BR(h_{SM} \to f)} , \]

where \( f \) is the final state of interest, and \( H \) is one of the two 125 GeV mass-degenerate scalars. The observed ratio of \( f \) production relative to the SM expectation is

\[ R_f \equiv \sum_H R_f^H . \]

In obtaining \( \sigma(pp \to S) \), we included the main Higgs production mechanisms: \( gg \) fusion, vector boson (\( W^+W^- \) and \( ZZ \)) fusion, and \( Vh \) production (\( V = W \) and \( Z \)). The final states of interest are \( f = \gamma\gamma, ZZ^*, WW^*, \tau^+\tau^- \) and \( b\bar{b} \).

In our analysis, we assumed that \( R_{WW} \simeq R_{ZZ} \simeq 1 \pm 0.2 \).
An enhanced $\gamma \gamma$ signal due to mass-degenerate $h^0$ and $A^0$:

Left panel: $R_{\gamma \gamma}$ as a function of $\tan \beta$ for $h$ (blue), $A$ (green), and the total observable rate (cyan), obtained by summing the rates with intermediate $h$ and $A$, for the unconstrained scenario.

Right panel: Total rate for $R_{\gamma \gamma}$ as a function of $\tan \beta$ for the constrained (red) and unconstrained (green) scenarios.

The enhancement occurs in the parameter regime of $\tan \beta \lesssim 1.5$.

An enhanced $\gamma \gamma$ signal in the mass-degenerate scenario yields two associated predictions that must be confirmed by experiment if this framework is to be consistent.
1. The inclusive $\tau^+\tau^-$ signal is enhanced with respect to the SM due to the production of $A$ via $gg$ fusion.

2. The exclusive $b\bar{b}$ signal due to the production of Higgs bosons in association with $W$ or $Z$ is close to its SM value but is not enhanced.

Left panel: Total $R_{\tau\tau}$ ($h$ and $A$ summed) as a function of $R_{\gamma\gamma}$ for the constrained (red) and unconstrained (green) scenarios. Right panel: $R_{bb}^{VH}$ ($h$ and $A$ summed) as a function of $R_{\gamma\gamma}$ for the constrained (red) and unconstrained (green) scenarios.
• The corresponding results in the Type-II 2HDM were similar. Other
degenerate-mass scalar pairs were also considered. By the end of Run I
of the LHC, the $\gamma\gamma$ excess was gone, and the Higgs data seemed to be
consistent with SM expectations.

• To my knowledge, there is no comprehensive analysis on the remaining
allowed regions of 2HDM parameter space with mass degenerate scalar
states based on the current LHC data sets.

• However, in this talk, I shall focus on some of the more theoretical aspects
of mass degeneracy in extended Higgs sectors. I shall first discuss the
theoretical basis for mass degeneracy in the 2HDM. I will then explore new
features that can arise if an additional Higgs doublet is present.
Any doublet extended Higgs model has a mass degenerate state—the charged Higgs boson, $H^\pm$. Indeed, $H^+$ and $H^-$ are degenerate due to the $U(1)_{EM}$ gauge symmetry. Moreover, the $H^+$ and $H^-$ are distinguishable by their electric charge, which we can probe using photons.

Suppose that this probe was unavailable (or equivalently, suppose one could turn off electromagnetism). Can experiment reveal the existence of a mass-degenerate scalar?

- Given a charged Higgs state, one could not physically distinguish between the two degenerate states.

- However, there would in principle be observables that are sensitive to the number of degenerate states present. Examples: $H \rightarrow H^+ H^-$ (but not $Z \rightarrow H^+ H^-$ due to the off-diagonal nature of this coupling).
Apart from the trivial mass-degeneracy of $H^{\pm}$, we would like to explore the possibility of mass-degenerate neutral scalars and/or mass-degenerate charged Higgs pairs. In each case, the critical questions to ask are:

- Is the origin of the mass degeneracy natural? (Yes, if due to a symmetry; No, if accidental)

- Can mass degenerate scalars be distinguished experimentally on an event by event basis?

- Is the only signal of the mass degeneracy a measurable multiplicity factor that can be determined when averaging over initial state degeneracies and summing over final state degeneracies?
Neutral scalar mass degeneracies in the 2HDM

Consider the 2HDM with hypercharge-one, doublet scalar fields $\Phi_1$ and $\Phi_2$. After minimizing the scalar potential, $\langle \Phi^0_i \rangle = v_i/\sqrt{2}$ (for $i = 1, 2$), where $|v_1|^2 + |v_2|^2 = (246 \text{ GeV})^2$ and $\tan \beta \equiv |v_2|/|v_1|$; the latter is basis-dependent and hence unphysical.

Introduce the Higgs basis fields,

$$H_1 = \begin{pmatrix} H^+_1 \\ H^0_1 \end{pmatrix} \equiv \frac{v_1^* \Phi_1 + v_2^* \Phi_2}{v}, \quad H_2 = \begin{pmatrix} H^+_2 \\ H^0_2 \end{pmatrix} \equiv \frac{-v_2 \Phi_1 + v_1 \Phi_2}{v},$$

such that $\langle H^0_1 \rangle = v/\sqrt{2}$ and $\langle H^0_2 \rangle = 0$. The Higgs basis is uniquely defined up to an overall rephasing, $H_2 \rightarrow e^{i\chi} H_2$.

We can immediately identify the physical charged Higgs field, $H^+ \equiv H^+_2$, and the neutral and charged Goldstone fields, $G^0 = \sqrt{2} \text{ Im } H^0_1$ and $G^+ \equiv H^+_1$. 
In the Higgs basis, the scalar potential is given by:

\[
V = Y_1 H_1^+ H_1 + Y_2 H_2^+ H_2 + [Y_3 H_1^+ H_2 + \text{h.c.}] + \frac{1}{2} Z_1 (H_1^+ H_1)^2 \\
+ \frac{1}{2} Z_2 (H_2^+ H_2)^2 + Z_3 (H_1^+ H_1)(H_2^+ H_2) + Z_4 (H_1^+ H_2)(H_2^+ H_1) \\
+ \left\{ \frac{1}{2} Z_5 (H_1^+ H_2)^2 + [Z_6 (H_1^+ H_1) + Z_7 (H_2^+ H_2)] H_1^+ H_2 + \text{h.c.} \right\},
\]

where \( Y_1, Y_2 \) and \( Z_{1,2,3,4} \) are real, whereas \( Y_3, Z_{5,6,7} \) are potentially complex.

After minimizing the scalar potential, \( Y_1 = -\frac{1}{2} Z_1 v^2 \) and \( Y_3 = -\frac{1}{2} Z_6 v^2 \).

**Remarks:**

1. Under the rephasing, \( H_2 \to e^{i\chi} H_2 \),

\[
[Y_3, Z_6, Z_7] \to e^{-i\chi}[Y_3, Z_6, Z_7] \quad \text{and} \quad Z_5 \to e^{-2i\chi} Z_5.
\]

2. Under the rephasing, \( H_2 \to e^{i\chi} H_2 \), the charged Higgs boson field is rephased, \( H^\pm \to e^{\pm i\chi} H^\pm \)

3. In the CP-conserving 2HDM, one can rephase the field \( H_2 \) such that all the parameters of the scalar potential are real.
Specializing to the Inert doublet model (IDM)

Suppose that the Higgs basis of the 2HDM exhibits an exact $\mathbb{Z}_2$ symmetry, $H_1 \to +H_1$ and $H_2 \to -H_2$. This symmetry is also preserved by the vacuum. It then follows that $Y_3 = Z_6 = Z_7 = 0$. The one remaining complex parameter, $Z_5$ can be chosen real by rephasing the Higgs basis field $H_2$. Thus, the IDM scalar potential is CP-conserving.

The Higgs basis doublet fields are also mass eigenstate fields,

$$H_1 = \left( \begin{array}{c} G^+ \\ \frac{1}{\sqrt{2}}[v + h + iG^0] \end{array} \right), \quad H_2 = \left( \begin{array}{c} H^+ \\ \frac{1}{\sqrt{2}}[H + iA] \end{array} \right),$$

where $G^\pm$ and $G^0$ are the Goldstone bosons that provide the longitudinal degrees of freedom of the massive $W^\pm$ and $Z^0$ gauge bosons. The tree-level properties of the scalar $h$ are precisely those of the SM Higgs boson. The physical scalar mass spectrum is,

$$m_h^2 = Z_1 v^2, \quad m_{H^\pm}^2 = Y_2 + \frac{1}{2}Z_3 v^2,$$

$$m_A^2 = m_{H^\pm}^2 + \frac{1}{2}(Z_4 - Z_5)v^2, \quad m_H^2 = m_A^2 + Z_5 v^2.$$
Scalar/vector Couplings of the IDM

\[ \mathcal{L}_{VVH} = \left( g_{mW} W_\mu^+ W^\mu - \frac{g}{2c_W} m_Z Z_\mu Z^\mu \right) h, \]

\[ \mathcal{L}_{VHH} = \left[ \frac{1}{4} g^2 W_\mu^+ W^\mu - \frac{g^2}{8c_W^2} Z_\mu Z^\mu \right] (h^2 + H^2 + A^2) \]

\[ + \left[ \frac{1}{2} g^2 W_\mu^+ W^\mu - e^2 A_\mu A^\mu + \frac{g^2}{c_W^2} \left( \frac{1}{2} - s_W^2 \right)^2 Z_\mu Z^\mu + \frac{2ge}{c_W} \left( \frac{1}{2} - s_W^2 \right) A_\mu Z^\mu \right] H^+ H^- \]

\[ + \left\{ \left( \frac{1}{2} eg A_\mu W_\mu^+ - \frac{g^2 s_W^2}{2c_W} Z^\mu W_\mu^\top \right) H^- (H + iA) + \text{h.c.} \right\}, \]

\[ \mathcal{L}_{VHH} = \frac{g}{2c_W} Z^\mu A_\mu \partial_\mu H - \frac{1}{2} g \left[ iW_\mu^+ H^- \partial_\mu (H + iA) + \text{h.c.} \right] \]

\[ + \left[ ie A_\mu + \frac{ig}{c_W} \left( \frac{1}{2} - s_W^2 \right) Z^\mu \right] H^+ \partial_\mu H^-, \]

where \( s_W \equiv \sin \theta_W, c_W \equiv \cos \theta_W \).

The trilinear and quadrilinear Higgs self-interactions are governed by

\[ \mathcal{L}_{3h} = -\frac{1}{2} v \left[ Z_1 h^3 + (Z_3 + Z_4) h (H^2 + A^2) + Z_5 h (H^2 - A^2) \right] - v Z_3 h H^+ H^- . \]

\[ \mathcal{L}_{4h} = -\frac{1}{8} \left[ Z_1 h^4 + Z_2 (H^2 + A^2)^2 + 2(Z_3 + Z_4) h^2 (H^2 + A^2) + 2Z_5 h^2 (H^2 - A^2) \right] \]

\[ -\frac{1}{2} H^+ H^- \left[ Z_2 (H^2 + A^2 + H^+ H^-) + Z_3 h^2 \right]. \]
Unnatural mass degeneracies of the IDM

1. \( m_H = m_h \), due to \( Z_1 v^2 = Y_2 + \frac{1}{2}(Z_3 + Z_4 + Z_5)v^2 \).

2. \( m_A = m_h \), due to \( Z_1 v^2 = Y_2 + \frac{1}{2}(Z_3 + Z_4 - Z_5)v^2 \).

Neither condition is a consequence of any symmetry and thus must be regarded as accidental (and unstable under radiative corrections).

However, even if such degeneracies exist, the degenerate states can be physically distinguished since \( h \) is even under the IDM \( \mathbb{Z}_2 \) symmetry, whereas \( H \) and \( A \) are odd. Thus, \( h \) can be produces by \( gg \) and \( VV \) fusion, whereas \( H \) and \( A \) must be produced via virtual \( Z \) exchange (the Drell-Yan process).
A natural mass degeneracy of the IDM

$m_H = m_A$, due to $Z_5 = 0$.

This mass degeneracy is due to an exact continuous $U(1)$ symmetry, $H_1 \to H_1$ and $H_2 \to e^{i\theta} H_2$, which is preserved by the vacuum. One can now define eigenstates of $U(1)$ charge,

$$\phi^\pm = \frac{1}{\sqrt{2}} [H \pm iA] .$$

The relevant interaction terms of $\phi^\pm$ are

$$\mathcal{L}_{\text{int}} = \left[ \frac{1}{2} g^2 W_{\mu}^+ W_{\mu}^- + \frac{g^2}{4 c_W^2} Z_\mu Z^\mu \right] \phi^+ \phi^- + \frac{ig}{2 c_W} Z_\mu \phi^- \partial^\mu \phi^+ - \frac{g}{\sqrt{2}} \left[ i W_{\mu}^+ H^- \partial^\mu \phi^+ + \text{h.c.} \right]$$

$$+ \frac{e g}{\sqrt{2}} \left( A^\mu W_{\mu}^+ H^- \phi^+ + A^\mu W_{\mu}^- H^+ \phi^- \right) - \frac{g^2 s_W^2}{\sqrt{2} c_W} \left( Z^\mu W_{\mu}^+ H^- \phi^+ + Z^\mu W_{\mu}^- H^+ \phi^- \right)$$

$$- v (Z_3 + Z_4) h \phi^+ \phi^- - \frac{1}{2} [Z_2 (\phi^+ \phi^-)^2 + (Z_3 + Z_4) h^2 \phi^+ \phi^-] - Z_2 H^+ H^- \phi^+ \phi^- .$$

Although $\phi^\pm$ are mass degenerate states, they can be physically distinguished.
For example, Drell-Yan production via a virtual $s$-channel $W^+$ exchange can produce $H^+$ in association with $\phi^-$, whereas virtual $s$-channel $W^-$ exchange can produce $H^-$ in association with $\phi^+$. Thus, the sign of the charged Higgs boson reveals the U(1)-charge of the produced neutral scalar. The origin of this correlation lies in the fact that, by construction, $H^+$ and $\phi^+$ both reside in $H_2$, whereas $H^-$ and $\phi^-$ both reside in $H_2^\dagger$.

**Remark:** The triply degenerate case, $m_h = m_H = m_A$ is also unnatural, as it requires the additional constraint, $Z_1 v^2 = Y_2 + \frac{1}{2} (Z_3 + Z_4)$. Even in this case, all the degenerate states are distinguishable, since $h$ is neutral under the U(1) symmetry. For example, Drell Yan production via $s$-channel virtual $W^-$ exchange can produce $H^- \phi^+$, but it cannot produce $H^- h$. 
Mass degeneracies in the most general 2HDM

To analyze the most general 2HDM, we note a remarkable tree-level relation

\[ \text{Im}(Z_5^*Z_6^2) = \frac{2s_{13}c_{13}s_{12}c_{12}}{v^2}(m_2^2 - m_1^2)(m_3^2 - m_1^2)(m_3^2 - m_2^2), \]

where the \( m_i \ (i = 1, 2, 3) \) are the masses of the three neutral Higgs bosons of the 2HDM, \( s_{12} \equiv \sin \theta_{12}, \ c_{12} \equiv \cos \theta_{12}, \) etc., and \( \theta_{12} \) and \( \theta_{13} \) are invariant mixing angles that are associated with the diagonalization of the neutral Higgs squared-mass matrix in the Higgs basis.

Thus, if any mass degeneracy is present, then one can find a Higgs basis in which \( Y_3, Z_5 \) and \( Z_6 \) are simultaneously real. Any CP-violating effects arise due to a potentially complex \( Z_7, \) which enters in the Higgs self-couplings but not the diagonalization of the tree-level neutral scalar squared-mass matrix.

Hence, without loss of generality, we can simply take \( Z_5 \) and \( Z_6 \) real and identify the neutral Higgs scalars as \( h, H \) and \( A. \) These are states of definite CP in their interactions with gauge bosons (and fermions).
The resulting Higgs mass relations are then,

\[
m^2_{H^\pm} = Y_2 + \frac{1}{2} Z_3 v^2, \quad m^2_A = m^2_{H^\pm} + \frac{1}{2} (Z_4 - Z_5)v^2,
\]

\[
m^2_{H,h} = \frac{1}{2} \left\{ m^2_A + (Z_1 + Z_5)v^2 \pm \sqrt{[m^2_A - (Z_1 - Z_5)v^2]^2 + 4Z_6^2v^4} \right\}.
\]

Mass degenerate states arise if one of the following two quantities is zero,

\[
Z_5(m^2_A - Z_1v^2) + Z_6^2v^2 = 0 \quad \text{or} \quad [m^2_A - (Z_1 - Z_5)v^2]^2 + 4Z_6^2v^4 = 0.
\]

**Case 1: \(m_h = m_H\)**

It follows that \(m^2_A = (Z_1 - Z_5)v^2\) and \(Z_6 = 0\). Thus, we recover the IDM mass spectrum for this degenerate case, although \(Z_7\) can be nonzero. Thus, the IDM scalar self couplings are modified by the addition of the following terms,

\[
\delta \mathcal{L}_{3h} = -\frac{1}{4} v [Z_7(H + iA) + Z_7^*(H - iA)] (HH + AA + 2H^+H^-),
\]

\[
\delta \mathcal{L}_{4h} = -\frac{1}{4} [Z_7(H + iA) + Z_7^*(H - iA)] (HH + AA + 2H^+H^-)h,
\]

which provide new sources of CP violation if \(\text{Im}\ Z_7 \neq 0\). The mass degeneracy is unnatural (moreover, \(Z_6 = 0\) is also unnatural when \(Z_7 \neq 0\)). Nevertheless, the mass-degenerate Higgs bosons are distinguishable as in the IDM.
Cases 2 and 3: $m_h = m_A$ or $m_H = m_A$

Both these possibilities arise when $Z_5(m_A^2 - Z_1v^2) + Z_6^2v^2 = 0$, which is an unnatural condition. The physical distinction of the mass degenerate states is due to the CP quantum numbers of the neutral scalar states (which are preserved in the tree-level Higgs interactions with gauge bosons and with fermions). One can therefore distinguish between the corresponding production mechanisms of the degenerate scalars that are mediated by gauge boson fusion or Drell-Yan production via $s$-channel gauge boson exchange.

Case 4: $m_h = m_H = m_A$

This requires $Z_5 = Z_6 = 0$ and $m_A^2 = Z_1v^2$. This leaves $Z_7$ as the only potentially complex parameter of the scalar potential in the Higgs basis, which can be chosen real by rephasing the Higgs basis field $H_2$. Hence, the Higgs scalar potential and vacuum must be CP-conserving. However, as long as $Z_7 \neq 0$, the triply mass-degenerate case is unnatural, since the $\mathbb{Z}_2$ symmetry of the IDM is not present.
New features of mass degenerate scalars in the 3HDM

In the 3HDM, one can now consider mass-degenerate charged Higgs pairs, as well as mass-degenerate neutral scalars. I will focus on two special 3HDMs where mass degeneracies occur.

The replicated IDM (RIDM)

We begin with a replicated IDM, in which the inert doublets are mass-degenerate. Consider the following 3HDM scalar potential in the Higgs basis,

\[
\mathcal{V}_{\text{RIDM}} = Y_1 H_1^\dagger H_1 + Y_2 \left( H_2^\dagger H_2 + H_3^\dagger H_3 \right) + \frac{1}{2} Z_1 (H_1^\dagger H_1)^2 + \frac{1}{2} Z_2 (H_2^\dagger H_2 + H_3^\dagger H_3)^2 \\
+ Z_3 (H_1^\dagger H_1) \left( H_2^\dagger H_2 + H_3^\dagger H_3 \right) + Z_4 \left[ (H_1^\dagger H_2)(H_2^\dagger H_1) + (H_1^\dagger H_3)(H_3^\dagger H_1) \right] \\
+ \frac{1}{2} Z_5 \left\{ (H_1^\dagger H_2)^2 + (H_2^\dagger H_1)^2 + (H_1^\dagger H_3)^2 + (H_3^\dagger H_1)^2 \right\}.
\]

Without loss of generality, we have chosen \( Z_5 \) real, so that \( \mathcal{V}_{\text{RIDM}} \) is CP-conserving. There is a continuous symmetry that is responsible for the mass-degeneracy of the inert Higgs doublets \( H_2 \) and \( H_3 \).
Consider the SU(2) family symmetry, where the doublet $H_1$ is a singlet and the doublets $H_2$ and $H_3$ transform as,

\[
\begin{pmatrix}
  H_2 \\
  H_3
\end{pmatrix} \rightarrow U \begin{pmatrix}
  H_2 \\
  H_3
\end{pmatrix}, \quad \text{with } U \in \text{SU}(2).
\]

If $Z_5 = 0$, then $\mathcal{V}_{\text{RIDM}}$ is invariant under SU(2). If $Z_5 \neq 0$, then $\mathcal{V}_{\text{RIDM}}$ is invariant under an SO(2) subgroup of the SU(2) transformations,

\[
\begin{pmatrix}
  H_2 \\
  H_3
\end{pmatrix} \rightarrow \begin{pmatrix}
  \cos \theta & \sin \theta \\
  -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
  H_2 \\
  H_3
\end{pmatrix}.
\]

The latter is enough to guarantee that the inert doublets $H_2$ and $H_3$ are mass degenerate. In the case of $Z_5 = 0$ (and the full SU(2) family symmetry), one has in addition a mass-degeneracy between the real and imaginary parts of each inert neutral scalar.
In the replicated IDM, the Higgs basis doublet fields are mass eigenstate fields,

\[ H_1 = \left( \frac{1}{\sqrt{2}} [G^+ + iG^0] \right), \quad H_2 = \left( \frac{1}{\sqrt{2}} [H + iA] \right), \quad H_3 = \left( \frac{1}{\sqrt{2}} [h + ia] \right), \]

with a minor change of notation from the IDM. The corresponding masses are,

\[ m_{H^\pm}^2 = m_{h^\pm}^2 = Y_2 + \frac{1}{2} Z_3 v^2, \quad m_H^2 = m_h^2 = Y_2 + \frac{1}{2} (Z_3 + Z_4 + Z_5) v^2, \]
\[ m_A^2 = m_a^2 = Y_2 + \frac{1}{2} (Z_3 + Z_4 - Z_5) v^2. \]

The corresponding couplings simply replicate the IDM couplings. For example,

\[ \mathcal{L}_{VVH} = \left( g m_W W^\mu_+ W^\mu_- + \frac{g}{2 c_W} m_Z Z_\mu Z^\mu \right) h_{SM}, \]
\[ \mathcal{L}_{VHH} = \frac{g}{2 c_W} Z^\mu (A \leftrightarrow \partial_\mu H + a \leftrightarrow \partial_\mu h) - \frac{1}{2} g \left[ i W^+_\mu H^{-} \leftrightarrow \partial_\mu (H + iA) + i W^+_\mu h^{-} \leftrightarrow \partial_\mu (h + ia) + \text{h.c.} \right] \]
\[ + \left[ i e A^\mu + \frac{ig}{c_W} \left( \frac{1}{2} - s^2_W \right) Z^\mu \right] (H^+ \leftrightarrow \partial_\mu H^- + h^+ \leftrightarrow \partial_\mu h^-), \]
\[ \mathcal{L}_{3h} = -\frac{1}{2} v \left[ Z_1 h_{SM}^3 + (Z_3 + Z_4) h_{SM}(H^2 + A^2 + h^2 + a^2) + Z_5 h_{SM}(H^2 - A^2 + h^2 - a^2) \right] \]
\[ - v Z_3 h_{SM}(H^+ H^- + h^+ h^-). \]
It is convenient to introduce,

\[ P \equiv \frac{H + ih}{\sqrt{2}}, \quad Q \equiv \frac{A - ia}{\sqrt{2}}, \]

where the extra minus sign is for later use. Then, we can rewrite the RIDM couplings in terms of the complex fields \( P \) and \( Q \). For example,

\[
\mathcal{L}_{VHH} = \frac{g}{2c_W} Z^\mu (Q \overleftrightarrow{\partial_\mu} P + Q^* \overleftrightarrow{\partial_\mu} P^*) - \frac{g}{2\sqrt{2}} \left[ (iW^+_\mu H^- - W^-_\mu h^+) \overleftrightarrow{\partial_\mu} (P + iQ) \\
- (iW^-_\mu H^- - W^+_\mu h^-) \overleftrightarrow{\partial_\mu} (P - iQ) + \text{h.c.} \right] \\
+ \left[ ieA_\mu + \frac{ig}{c_W} (\frac{1}{2} - s^2_W) Z^\mu \right] (H^+ \overleftrightarrow{\partial_\mu} H^- + h^+ \overleftrightarrow{\partial_\mu} h^-),
\]

\[
\mathcal{L}_{3h} = -v \left[ \frac{1}{2} Z_1 h^3_{\text{SM}} + (Z_3 + Z_4) h_{\text{SM}} (|P|^2 + |Q|^2) + Z_5 h_{\text{SM}} (|P|^2 - |Q|^2) \right] - v Z_3 h_{\text{SM}} (H^+ H^- + h^+ h^-).
\]

In the RIDM, there is no experimental measurement that can physically distinguish the degenerate scalars, \((H^\pm, h^\pm), (H, h)\) and \((A, a)\). However, the multiplicity factor will appear after summing over final mass-degenerate states, e.g., \(Z \to HA, ha\) (or equivalently, \(Z \to PQ, P^*Q^*\)), doubles the rate into a pair of neutral scalars.
The Ivanov-Silva Model

Ivanov and Silva (IS) introduced a particular 3HDM model with some curious properties.† In the Higgs basis of the 3HDM, we are free to make an arbitrary U(2) rotation to define the Higgs basis fields, $H_2$ and $H_3$. We have made use of this freedom to make a minor alteration of the IS scalar potential,

$$V_{IS} = V_{R IDM} + Z_3'(H_2^\dagger H_2)(H_3^\dagger H_3) + Z_4'(H_2^\dagger H_3)(H_3^\dagger H_2)$$

$$+ \left[ Z_8(H_2^\dagger H_3)^2 + Z_9(H_2^\dagger H_3)(H_2^\dagger H_2 - H_3^\dagger H_3) + \text{h.c.} \right],$$

where $V_{R IDM}$ is the replicated IDM scalar potential. Note that after the extra terms above are included, there is no remaining unbroken continuous subgroup of the SU(2) family symmetry (which was responsible for the two mass-degenerate inert doublets of the RIDM).

Nevertheless, the Ivanov-Silva model still yields two mass-degenerate inert doublets, since none of the extra terms involve the Higgs basis field $H_1$. Hence, these terms do not contribute to the tree-level scalar squared-mass matrices.

†I.P. Ivanov and J.P. Silva, Phys. Rev. D 93, 095014 (2016) [arXiv:1512.09276],
That is, the Higgs basis doublet fields of the Ivanov-Silva model are mass eigenstate fields,

\[
H_1 = \left(\frac{1}{\sqrt{2}}[G^+ v + h_{\text{SM}} + iG^0]\right), \quad H_2 = \left(\frac{1}{\sqrt{2}}[H + iA]\right), \quad H_3 = \left(\frac{1}{\sqrt{2}}[h + ia]\right),
\]

where the corresponding scalar squared-masses are the same as in the RIDM,

\[
m_{H^\pm}^2 = m_{h^\pm}^2 = Y_2 + \frac{1}{2}Z_3v^2, \quad m_H^2 = m_h^2 = Y_2 + \frac{1}{2}(Z_3 + Z_4 + Z_5)v^2, \\
m_A^2 = m_a^2 = Y_2 + \frac{1}{2}(Z_3 + Z_4 - Z_5)v^2.
\]

Is the mass degeneracy of the two inert doublets natural? One can check that \(\mathcal{V}_{\text{RIDM}}\) is invariant under a discrete \(\mathbb{Z}_4\) subgroup of the SU(2) family symmetry group. The elements of this subgroup are,

\[
\mathbb{Z}_4 = \{I, -I, Z, -Z\}, \quad \text{where } Z \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

where the \(2 \times 2\) matrices above act on the Higgs basis fields \(H_2\) and \(H_3\). Note that \(Z^2 = -I\), where \(I\) is the \(2 \times 2\) identity matrix.
The fields $H_2$ and $H_3$ are odd under $-I$, which simply identifies the two inert doublets. The elements $Z$ (and $-Z$) act non-trivially on the inert doublets. As before, we are free to combine mass-degenerate neutral fields and define,

$$P \equiv (H + ih)/\sqrt{2} \quad \text{and} \quad Q \equiv (A - ia)/\sqrt{2},$$

which are eigenstates of $Z$ (and $-Z$). Indeed, $P$ and $Q^*$ have eigenvalue $i$ under $Z$, where $P^*$ and $Q$ have eigenvalue $-i$ under $Z$. For example, this is consistent with the couplings of neutral scalars to the $Z$, namely

$$\mathcal{L}_{ZHH} = \frac{g}{2c_W} Z^\mu (P \overset{\leftrightarrow}{\partial}_\mu Q + P^* \overset{\leftrightarrow}{\partial}_\mu Q^*).$$

Thus, it appears that the discrete symmetry group $\mathbb{Z}_4$ is sufficient, due to invariance of $\mathcal{V}_{IS}$ with respect to

$$\begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_2 \\ H_3 \end{pmatrix},$$

to preserve the mass degeneracy of the two inert doublets!
**CP4 symmetry of the Ivanov-Silva Model**

However, there is another symmetry at play here. Ivanov and Silva showed that there exists a generalized CP (GCP) transformation,

\[ H_i \rightarrow X_{ij} H_j^* , \quad \text{where } X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \]

under which \( V_{IS} \) is invariant. Applying this transformation twice yields \( \text{diag}(1, -1, -1) \) so one must apply it four times to obtain the identity. Ivanov and Silva call this transformation CP4, in contrast to ordinary CP whose square is the identity. Perhaps the CP4 symmetry is responsible for preserving the mass degeneracy of the two inert doublets.

It is straightforward to check that the fields \( P, Q \) and the complex conjugates are eigenstates of CP4. That is, under CP4, \( P \rightarrow iP, Q \rightarrow iQ, P^* \rightarrow -iP^*, \) and \( Q^* \rightarrow -iQ^* \). Indeed, \((\text{CP4})^4 = I\), as required.
The CP4 quantum numbers of the neutral scalars are consistent with their couplings to the $Z$; namely

$$\mathcal{L}_{ZHH} = \frac{g}{2c_W} Z^\mu (Q \overset{\leftrightarrow}{\partial_\mu} P + Q^* \overset{\leftrightarrow}{\partial_\mu} P^*) ,$$

since $QP$ and $Q^*P^*$ change sign under CP4, whereas the $Z$ is CP-even and the derivative operator yields an extra minus sign under $P$.

The Ivanov-Silva model is CP-conserving due to the CP4 symmetry, even though the conventionally defined CP symmetry, $H_i \rightarrow H_i^*$, which satisfies $(\text{CP})^2 = I$, is not a symmetry. In particular, note that the parameters $Z_8$ and $Z_9$ of $\mathcal{V}_{IS}$ are potentially complex (in a Higgs basis where $Z_5$ is real). If initially complex, no basis change involving $H_2$ and $H_3$ exists in which both $Z_8$ and $Z_9$ are real.‡

‡Contrast this with a GCP-invariant 2HDM. In this case, $H_i \rightarrow H_i^*$ is automatically a symmetry, which implies that a Higgs basis exists in which all scalar potential parameters are real and the conventionally defined CP symmetry is conserved. This is no longer true in the NHDM with $N > 2$. 
Four critical questions still unanswered

1. Is there any connection between the degenerate scalar spectrum of the Ivanov-Silva (IS) model and the observation that there is no real Higgs basis despite the presence of the CP4 symmetry?

   After all, it seems that one can argue that the CP4 symmetry is responsible for the degenerate Higgs spectrum.

2. Is there any observable distinction between CP4-invariant models and conventionally CP-invariant models? (Ivanov says NO.)

3. Can any of the mass-degenerate scalars of the IS model be experimentally distinguished (apart from the multiplicity factor in observables that sum over degenerate final states)?

4. Is there a mapping between the IS model and an extended Higgs model with conventional CP properties with the same degenerate scalar spectrum that would show that the two models are experimentally indistinguishable?

   It is our hope that we can at least answer the last two questions in the very near future.